

# QUASITRIANGULAR COIDEAL SUBALGEBRAS OF $U_q(\mathfrak{g})$ IN TERMS OF GENERALIZED SATAKE DIAGRAMS

VIDAS REGELSKIS AND BART VLAAR

ABSTRACT. Let  $\mathfrak{g}$  be a finite-dimensional semisimple complex Lie algebra and  $\theta$  an involutive automorphism of  $\mathfrak{g}$ . It is well-known from works of Letzter, Kolb and Balagović that the fixed-point subalgebra  $\mathfrak{k} = \mathfrak{g}^\theta$  has a quantum counterpart  $B$ , a coideal subalgebra of the Drinfeld-Jimbo quantum group  $U_q(\mathfrak{g})$  possessing a cylinder-twisted universal  $K$ -matrix  $\mathcal{K}$ . The objects  $\theta$ ,  $\mathfrak{k}$ ,  $B$  and  $\mathcal{K}$  can all be described in terms of a combinatorial datum, a Satake diagram. In the present work we extend this construction to generalized Satake diagrams, objects first considered by Heck. A generalized Satake diagram defines a semisimple automorphism of  $\mathfrak{g}$  restricting to the standard Cartan subalgebra  $\mathfrak{h}$  as an involution. We show that it naturally leads to a subalgebra  $\mathfrak{k} \subset \mathfrak{g}$ , not necessarily a fixed-point subalgebra, but still satisfying  $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta$ . Such a subalgebra  $\mathfrak{k}$  can be quantized to a coideal subalgebra of  $U_q(\mathfrak{g})$  endowed with a cylinder-twisted universal  $K$ -matrix. We conjecture that all such coideal subalgebras of  $U_q(\mathfrak{g})$  arise from generalized Satake diagrams in this way.

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## 1. INTRODUCTION

Given a finite-dimensional semisimple complex Lie algebra  $\mathfrak{g}$  and an involutive Lie algebra automorphism  $\theta \in \text{Aut}(\mathfrak{g})$ , a *symmetric pair* is a pair  $(\mathfrak{g}, \mathfrak{k})$  where  $\mathfrak{k} = \mathfrak{g}^\theta$  is the  $\theta$ -fixed subalgebra of  $\mathfrak{g}$ , see [Ara62, Sat71]. *Quantum symmetric pairs* are their quantum analogons. That is to say, the enveloping algebra  $U(\mathfrak{g})$  can be quantized to a quasitriangular Hopf algebra, the Drinfeld-Jimbo quantum group  $U_q(\mathfrak{g})$  endowed with the universal  $R$ -matrix  $\mathcal{R}$ , see [Ji85, Dr87]. Similarly, the  $\theta$ -fixed subalgebra  $\mathfrak{k}$  can be quantized to a coideal subalgebra  $B \subseteq U_q(\mathfrak{g})$  [Let99, Let02, Ko14] having a compatible quasitriangular structure, the cylinder-twisted universal  $K$ -matrix  $\mathcal{K}$  [BK16, Ko17].

The involution  $\theta$ , the corresponding fixed-point subalgebra  $\mathfrak{k}$ , the coideal subalgebra  $B$  and the universal object  $\mathcal{K}$  are all defined in terms of a combinatorial data, the so-called Satake diagram  $(X, \tau)$ . Here  $X$  is a subdiagram of the Dynkin diagram of  $\mathfrak{g}$  and  $\tau$  is an involutive diagram automorphism stabilizing  $X$  and satisfying certain compatibility conditions, see [Let02, Ko14].

It is the aim of this paper to extend some of the above work to a more general setting than (quantizations of) fixed-point subalgebras. A direct motivation for this is the fact that the correct quantum group analogue of the fixed-point subalgebra in the Letzter-Kolb approach is not a fixed-point subalgebra itself, but merely tends to one as  $q \rightarrow 1$ , see [Ko14, Ch. 10]. This suggests that there may be a generalization of this approach that does not require a fixed-point subalgebra as input.

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A careful analysis of [Ko14, BK15, BK16] indeed indicates that the compatibility conditions for  $X$  and  $\tau$  can be weakened, leading to the notion of a *generalized Satake diagram*, see Definition 2.2, and the whole theory survives in this setting with very minor adjustments. The resulting Lie subalgebra  $\mathfrak{k} = \mathfrak{k}(X, \tau)$  is given in Definition 3.1 and the corresponding coideal subalgebra  $B = B(X, \tau)$  in Definition 4.1. Indeed, in [BK15, Rmks. 2.6, 3.14] it is explicitly suggested that some key passages of the theory are amenable for generalizations.

Our proposed generalization of Satake diagrams can be traced back to the work of A. Heck [He84]. In this work Heck provides a classification of involutions of finite root systems such that the corresponding restricted Weyl group is the Weyl group of the restricted root system. We will review this point-of-view and make a connection with a theorem of Lusztig stating that the restricted Weyl group is in fact a Coxeter group.

The characterization in terms of the restricted Weyl group is relevant in the context of the universal  $R$ - and  $K$ -matrices for quantum symmetric pairs. The universal  $R$ -matrix  $\mathcal{R}$  has a distinguished factor called quasi  $R$ -matrix playing an important role in the theory of canonical bases for  $U_q(\mathfrak{g})$  developed by Kashiwara and Lusztig, see [Ka90] and [Lu94, Part IV]. This object possesses a remarkable factorization property expressed in terms of the braid group action on  $U_q(\mathfrak{g})$  of the Weyl group associated to  $\mathfrak{g}$ , see e.g. [KR90, LS90]. Recently it has become clear that many of these properties extend to the cylinder-twisted universal  $K$ -matrix  $\mathcal{K}$ . It has a distinguished factor called quasi  $K$ -matrix introduced in [BW13] for certain coideal subalgebras of  $U_q(\mathfrak{sl}_N)$  and in a more general setting in [BK15], and featuring prominently in the theory of canonical bases for quantum symmetric pairs [BW16]. In [DK18] a factorization property is established for the quasi  $K$ -matrix using a braid group action of the aforementioned restricted Weyl group. In the present work we argue that the factorization property extends to quasi  $K$ -matrices defined in terms of the generalized Satake diagrams.

A generalization of this approach to the Kac-Moody setting will be addressed in a future work. Another outstanding issue is a Lie-theoretic motivation of the subalgebra  $\mathfrak{k}$ , which we define in a rather *ad hoc* manner directly in terms of the combinatorial data  $(X, \tau)$ , see Definition 3.1.

Therefore let us end the introduction with an additional motivation for the study of the subalgebra  $\mathfrak{k}$  and its quantization  $B$  by making some observations related to the representation theory of the pair  $(U_q(\mathfrak{g}), B)$ . Following [BK16, Ko17], there exists a suitable completion  $\mathcal{U}$  of  $U_q(\mathfrak{g})$  such that the objects  $\mathcal{R} \in (\mathcal{U} \otimes \mathcal{U})^\times$  and  $\mathcal{K} \in \mathcal{U}^\times$  have well-defined images under any finite-dimensional representation  $\rho : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$ . Furthermore, there exists an involutive Hopf algebra automorphism  $\phi$  of  $\mathcal{U}$  such that  $(\phi \otimes \phi)(\mathcal{R}) = \mathcal{R}$  and the following quartic relation is satisfied, known as the (*universal*)  $\phi$ -*twisted reflection equation* (see [Ko17, Eqs. (3.22-3.23)]):

$$(1.1) \quad \mathcal{R}_{21} \mathcal{K}_2 (\phi \otimes \text{id})(\mathcal{R}) \mathcal{K}_1 = \mathcal{K}_1 (\phi \otimes \text{id})(\mathcal{R}_{21}) \mathcal{K}_2 \mathcal{R} \quad \in \mathcal{U} \otimes \mathcal{U}$$

where  $\mathcal{K}_1 = \mathcal{K} \otimes 1$ ,  $\mathcal{K}_2 = 1 \otimes \mathcal{K}$ ,  $\mathcal{R}_{21} = \sigma(\mathcal{R})$  and  $\sigma \in \text{Aut}_{\text{alg}}(\mathcal{U} \otimes \mathcal{U})$  is the flip map. Let  $R \in \text{GL}(V \otimes V)$  be proportional to  $(\rho \otimes \rho)(\mathcal{R})$  and  $K \in \text{GL}(V)$  proportional to  $\rho(\mathcal{K})$ . In the case  $\phi = \text{id}$ , applying  $\rho \otimes \rho$  to (1.1) one obtains the matrix reflection equation

$$(1.2) \quad R_{21} K_2 R K_1 = K_1 R_{21} K_2 R \quad \in \text{End}(V \otimes V)$$

where  $K_1 = K \otimes \text{Id}$ ,  $K_2 = \text{Id} \otimes K$  and  $R_{21} = PRP$  with  $P : V \otimes V \rightarrow V \otimes V$  the permutation operator. When  $\phi \neq \text{id}$  one naturally obtains the so-called twisted matrix reflection equation which we omit for simplicity, but this does not significantly affect any of the following remarks. In particular, starting with a Satake diagram, one will recover the solutions of (1.2) used in [NDS95, NS95] to define quantum symmetric pairs.

Treating the matrix  $R$  as given, one can of course solve (1.2) for  $K \in \text{GL}(V)$ . For  $U_q(\mathfrak{sl}_N)$  and  $V = \mathbb{C}^N$  this was done by A. Mudrov [Mu02]. Based on this result and computations for  $U_q(\mathfrak{g})$  with  $\mathfrak{g}$  of types  $B_n$ ,  $C_n$ ,  $D_n$  ( $n \leq 4$ ) and  $G_2$ , and  $V$  the vector representation, we formulate the following conjecture.

**Conjecture 1.1.** *Let  $\rho : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$  be the vector representation of  $U_q(\mathfrak{g})$ . If  $K \in \text{GL}(V)$  is a solution of (1.2) then there exists a generalized Satake diagram  $(X, \tau)$  such that  $K$  is proportional to  $\rho(\mathcal{K})$  where  $\mathcal{K}$  is the universal  $K$ -matrix for the coideal subalgebra  $B(X, \tau)$ , i.e. the quantization of  $U(\mathfrak{k}(X, \tau))$ .*

Based on the available evidence in terms of solutions to (1.2) known to intertwine restrictions of  $\rho$  to coideal subalgebras, we also make the following claim.

**Conjecture 1.2.** *Let  $\rho : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$  be the vector representation of  $U_q(\mathfrak{g})$ . Then  $\rho$  can be used to identify coideal subalgebras, i.e. if the distinct coideal subalgebras  $B, B' \subseteq U_q(\mathfrak{g})$  possess the universal  $K$ -matrices  $\mathcal{K}$  and  $\mathcal{K}'$ , respectively, then  $\rho(\mathcal{K})$  and  $\rho(\mathcal{K}')$  are not scalar multiples of each other.*

If these two conjectures are true, the only coideal subalgebras of  $U_q(\mathfrak{g})$  which possess a universal  $K$ -matrix in the sense of [Ko17] are those which are quantizations of  $U(\mathfrak{k}(X, \tau))$  with  $(X, \tau)$  a generalized Satake diagram.

We should remark that coideal subalgebras  $B$  in the Letzter-Kolb approach carry additional parameters. The generators associated to the nodes  $i \in I \setminus X$  depend on scalars  $\gamma_i \neq 0$  and  $s_i$ , see Definition 4.1. We can thus sharpen Conjecture 1.1. Any invertible matrix solution  $K$  of (1.2) is proportional to  $\rho(\mathcal{K})$  for some  $B(X, \tau)$  with the additional parameters satisfying certain constraints. Most of these constraints were found in [Let03, Ko14] given in terms of the sets  $\Gamma_q$  and  $\mathcal{S}_q$ , see (4.3). Always, we must have  $(\gamma_i)_{i \in I \setminus X} \in \Gamma_q$ . For the conditions on  $s_i$  it is helpful to consider the set  $I_{\text{ns}} = \{i \in I \setminus X \mid i \text{ does not neighbour } X, \tau(i) = i\}$ , see (3.16). The constraints on the  $s_i$  are as follows. If  $i \notin I_{\text{ns}}$  then  $s_i = 0$ . For all  $(i, j) \in I_{\text{ns}} \times I_{\text{ns}}$  such that  $i \neq j$  conjecturally one of three conditions must hold: the Cartan integer  $a_{ij}$  is even,  $s_j = 0$ , or  $s_i^2/\gamma_i$  lies in a particular finite subset of a quadratic completion of  $\mathbb{C}(q)$ . The defining condition of the set  $\mathcal{S}_q$  does not cover the third possibility, which appeared in [BB10].

The paper is organized as follows. Section 2 contains the preliminaries and basic definitions. We define the necessary Lie-theoretic objects surrounding a finite-dimensional semisimple complex Lie algebra  $\mathfrak{g}$  and its Cartan subalgebra  $\mathfrak{h}$ . We introduce the notion of a generalized Satake diagram as a decoration of the Dynkin diagram of  $\mathfrak{g}$ . We explain how the generalized Satake diagrams emerge in the work of A. Heck.

In Section 3 we define the main object of this paper, the subalgebra  $\mathfrak{k} = \mathfrak{k}(X, \tau) \subseteq \mathfrak{g}$ . Theorem 3.2 is the main result of this section. We show that  $\mathfrak{k}$  satisfies the intersection condition  $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta$  (which trivially holds when  $\mathfrak{k} = \mathfrak{g}^\theta$  with  $\theta^2 = \text{id}_{\mathfrak{g}}$ ) precisely if  $(X, \tau)$  is a generalized Satake diagram. We then study the derived subalgebra of  $\mathfrak{k}$ . When  $\mathfrak{k}$  is not a reductive Lie algebra, Propositions 3.5 and 3.6 establish a semidirect product decomposition for  $\mathfrak{k}$  in terms of a reductive subalgebra and a nilpotent ideal of class 2. We end this section with some results about the universal enveloping algebra  $U(\mathfrak{k})$ . (Appendix A contains three technical lemmas in aid of Section 3.)

In Section 4 we briefly review the quasitriangular structure behind the quantum symmetric pairs. We indicate the necessary modifications to the theory of Balagović-Kolb so that it would be applicable to the quantum pair algebras associated to the generalized Satake diagrams.

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## 2. FINITE-DIMENSIONAL SEMISIMPLE LIE ALGEBRAS AND ROOT SYSTEM INVOLUTIONS

Let  $I$  be a finite set and  $A = (a_{ij})_{i, j \in I}$  a Cartan matrix. In particular, there exist positive rationals  $d_i$  ( $i \in I$ ) such that  $d_i a_{ij} = d_j a_{ji}$ . Let  $\mathfrak{g} = \mathfrak{g}(A)$  be the corresponding finite-dimensional semisimple Lie algebra over  $\mathbb{C}$ . More precisely,  $\mathfrak{g}$  is generated by  $\{e_i, f_i, h_i\}_{i \in I}$  subject to

$$(2.1) \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i$$

$$(2.2) \quad \text{ad}(e_i)^{1-a_{ij}}(e_j) = \text{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{if } i \neq j,$$

for all  $i, j \in I$ . We denote the standard Cartan subalgebra by  $\mathfrak{h} = \langle h_i \mid i \in I \rangle$  and also consider the corresponding nilpotent subalgebras  $\mathfrak{n}^+ = \langle e_i \mid i \in I \rangle$ ,  $\mathfrak{n}^- = \langle f_i \mid i \in I \rangle$ .

The simple roots  $\alpha_i \in \mathfrak{h}^*$  ( $i \in I$ ) satisfy  $\alpha_j(h_i) = a_{ij}$  for  $i, j \in I$ . Let  $Q = \sum_{i \in I} \mathbb{Z} \alpha_i$  denote the root lattice. In terms of the root spaces  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$  ( $\alpha \in Q$ ),  $\mathfrak{g}$  is a  $Q$ -graded Lie

algebra and we have the following identities for  $\mathfrak{h}$ -modules:

$$(2.3) \quad \mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in Q^+} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{h} = \mathfrak{g}_0.$$

Hence the root system  $\Phi := \{\alpha \in Q \mid \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0\}$  satisfies  $\Phi = \Phi^+ \cup \Phi^-$  where  $\Phi^\pm = \pm(\Phi \cap Q^+)$  and  $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ .

The Weyl group  $W$  is a finite subgroup of  $\mathrm{GL}(\mathfrak{h}^*)$  generated by the simple reflections  $r_i$  ( $i \in I$ ) acting via  $r_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$  for all  $i \in I$ ,  $\alpha \in \mathfrak{h}^*$ . More precisely,  $W$  is a normal subgroup of

$$\mathrm{Aut}(\Phi) := \{g \in \mathrm{GL}(\mathfrak{h}^*) \mid g(\Phi) = \Phi\}.$$

Since  $W$  induces a simple transitive action on the set of bases of  $\Phi$ , one readily obtains that  $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(A)$ , where

$$\mathrm{Aut}(A) = \{\sigma : I \rightarrow I \text{ invertible} \mid a_{\sigma(i)\sigma(j)} = a_{ij} \text{ for all } i, j \in I\}$$

is the group of diagram automorphisms (acting by relabelling).

The following subgroup of  $\mathrm{Aut}(\mathfrak{g})$  will be important in what follows:

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) = \{\sigma \in \mathrm{Aut}(\mathfrak{g}) \mid \sigma(\mathfrak{h}) = \mathfrak{h}\} < \mathrm{Aut}(\mathfrak{g}).$$

We briefly review some important subgroups of  $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ . A braid group action on  $\mathfrak{g}$  which extends the dual action of  $W$  on  $\mathfrak{h}$  is defined by  $\mathrm{Ad}(r_i) = \exp(\mathrm{ad}(e_i)) \exp(\mathrm{ad}(-f_i)) \exp(\mathrm{ad}(e_i)) \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$  for  $i \in I$ , yielding  $\mathrm{Ad}(W) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ . We also have  $\mathrm{Aut}(A) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$  (acting by relabelling). The Chevalley involution  $\omega \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$  is defined by swapping  $e_i$  and  $-f_i$  for all  $i \in I$ ; it commutes with  $\mathrm{Ad}(W)$  and with  $\mathrm{Aut}(A)$ . Finally, the group  $\tilde{H} := \mathrm{Hom}(Q, \mathbb{C}^\times)$  naturally induces a subgroup  $\mathrm{Ad}(\tilde{H}) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$  via  $\mathrm{Ad}(\chi)|_{\mathfrak{g}_\alpha} = \chi(\alpha) \mathrm{id}_{\mathfrak{g}_\alpha}$  for all  $\chi \in \tilde{H}$ ,  $\alpha \in Q$ .

The elements of  $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$  can be dualized to elements of  $\mathrm{Aut}(\Phi)$ . Conversely, given  $g \in \mathrm{Aut}(\Phi)$  there are  $\psi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$  whose restriction to  $\mathfrak{h}$  dualizes to  $g$ . Indeed, from  $-\mathrm{id}_{\mathfrak{h}^*} \in \mathrm{Aut}(\Phi)$  and the direct product decomposition  $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(A)$ , there exist unique  $(w, \tau) \in W \times \mathrm{Aut}(A)$  such that  $g = -w\tau$ . Then one easily checks that  $\psi = \mathrm{Ad}(w)\omega\tau \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$  satisfies  $(\psi|_{\mathfrak{h}})^* = g$ .

**2.1. Compatible decorations and involutions of  $\Phi$ .** Given a subset  $X \subseteq I$  denote the corresponding Cartan submatrix by  $A_X = (a_{ij})_{i,j \in X}$  and consider the corresponding semisimple Lie algebra  $\mathfrak{g}_X := \langle e_i, f_i, h_i \mid i \in X \rangle \subseteq \mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h}_X = \mathfrak{h} \cap \mathfrak{g}_X$ , dual Weyl vector  $\rho_X^\vee \in \mathfrak{h}_X$  and Weyl group  $W_X := \langle r_i \mid i \in X \rangle \leq W$ . The unique longest element  $w_X \in W_X$  is an involution and there exists  $\tau_{0,X} \in \mathrm{Aut}(A_X)$  which satisfies

$$(2.4) \quad -w_X(\alpha_i) = \alpha_{\tau_{0,X}(i)} \quad \text{for all } i \in X.$$

Note that  $\mathrm{Ad}(w_X)|_{\mathfrak{g}_X} = \tau_{0,X}\omega|_{\mathfrak{g}_X}$  and  $\mathrm{Ad}(w_X)^2|_{\mathfrak{g}_\alpha} = \zeta(\alpha) \mathrm{id}_{\mathfrak{g}_\alpha}$  for all  $\alpha \in \Phi$ , where  $\zeta = \zeta(X) \in \tilde{H}$  is defined by

$$\zeta(\alpha_i) := (-1)^{2\alpha_i(\rho_X^\vee)} \quad \text{for } i \in I.$$

We will study

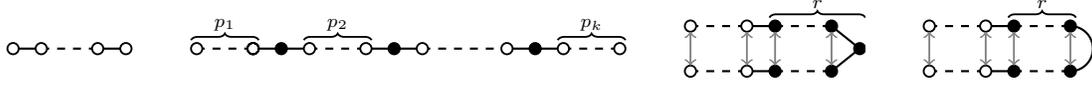
$$\begin{aligned} \mathrm{Aut}^{\mathrm{inv}}(\mathfrak{g}, \mathfrak{h}) &:= \{\psi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) \mid \psi^2|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\}, \\ \mathrm{Aut}^{\mathrm{inv}}(\Phi) &:= \{g \in \mathrm{Aut}(\Phi) \mid g^2 = \mathrm{id}_{\mathfrak{h}^*}\} \end{aligned}$$

by means of combinatorial data: we define

$$(2.5) \quad \mathrm{CDec}(A) = \{(X, \tau) \mid X \subseteq I, \tau \in \mathrm{Aut}(A), \tau^2 = \mathrm{id}_I, \tau(X) = X, \tau|_X = \tau_{0,X}\}$$

and call its elements *compatible decorations* (of  $A$ ). In the Dynkin diagram associated to  $\mathfrak{g}$  one marks this decoration by filling the nodes corresponding to  $X$  and drawing two-sided arrows for the nontrivial orbits of  $\tau$ .

**Example 2.1.** Let  $A$  be of type  $A_n$ ,  $n \geq 2$ . The compatible decorations  $\text{CDec}(A)$  are



where  $p_1, p_k \in \mathbb{Z}_{\geq 0}$ ,  $p_2, \dots, p_{k-1} \in \mathbb{Z}_{\geq 1}$  for any  $k \in \mathbb{Z}_{\geq 2}$  and  $0 \leq r \leq \lceil n/2 \rceil$ .

Given  $(X, \tau) \in \text{CDec}(A)$ , we define

$$(2.6) \quad \theta = \theta(X, \tau) = -w_X \tau \in \text{Aut}^{\text{inv}}(\Phi).$$

As explained above, the map dual to  $\theta$  can be extended to an element of  $\text{Aut}^{\text{inv}}(\mathfrak{g}, \mathfrak{h})$  which we shall also call  $\theta$ . It is given by  $\theta = \text{Ad}(w_X)\tau\omega$  so that  $\theta|_{\mathfrak{h}} = -w_X\tau$ . Note that, as a consequence of properties of  $\text{Ad}(w_X)$  mentioned earlier, we have

$$(2.7) \quad \theta|_{\mathfrak{g}_X} = \text{id}_{\mathfrak{g}_X},$$

$$(2.8) \quad \theta^2|_{\mathfrak{g}_\alpha} = \zeta(\alpha)\text{id}_{\mathfrak{g}_\alpha} \quad \text{for all } \alpha \in \Phi.$$

**2.2. Generalized Satake diagrams and the restricted Weyl group.** We choose a subset  $I^* \subseteq I \setminus X$  such that it contains precisely one element from each  $\tau$ -orbit in  $I \setminus X$ . For  $i \in I^*$  denote by  $\check{X}(i) \subseteq X$  the union of connected components of  $X$  neighbouring  $\{i, \tau(i)\}$  and  $\check{X}[i] := \check{X}(i) \cup \{i, \tau(i)\}$ . By a *minimal subdiagram* of  $(X, \tau) \in \text{CDec}(A)$  we mean any subdiagram of the form  $\check{X}[i]$  for some  $i \in I^*$ . By definition  $\check{X}[i]$  is a compatible decoration of  $A_{\check{X}[i]}$ ; it is also known as a Satake diagram of (restricted) rank 1.

**Definition 2.2.** *Generalized Satake diagrams* are elements of the set

$$\text{GSat}(A) := \{(X, \tau) \in \text{CDec}(A) \mid (X, \tau) \text{ contains no minimal subdiagram of the form } \circ \bullet\}. \quad \emptyset$$

The compatible decorations in Example 2.1 are generalized Satake diagrams when  $p_1 = p_k = 0$  and  $p_2 = \dots = p_{k-1} = 1$ .

*Remark 2.3.* Generalized Satake diagrams were first considered by Heck in [He84] where they are shown to classify involutions of root systems such that the restricted Weyl group is the Weyl group of the restricted root system. Heck uses the symbol  $\sigma$  to denote the negative of our map  $\theta$ . He also uses the term Satake diagram for any  $(X, \tau)$  such that  $X \subseteq I$ ,  $\tau \in \text{Aut}(A)$ ,  $\tau^2 = \text{id}_I$  and  $\tau(X) = X$  (this properly contains the set  $\text{CDec}(A)$ ) and the elements of  $\text{GSat}(A)$  are called admissible Satake diagrams. However, the term Satake diagram has become reserved for those combinatorial data which classify involutions of  $\mathfrak{g}$  up to conjugacy (and their fixed-point subalgebras), which is the reason for our nomenclature ‘‘compatible decoration’’ and ‘‘generalized Satake diagram’’.  $\emptyset$

Note that  $(X, \tau)$  is a generalized Satake diagrams precisely if

$$(2.9) \quad \forall (i, j) \in I \setminus X \times X : \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j \implies a_{ij} \neq -1,$$

which is precisely the condition needed in [Ko14, Proof of Lemma 5.11, Step 1] and [BK16, Proof of Lemma 6.4]. One can show that (2.9) is equivalent to either of the following more compact conditions:

$$\begin{aligned} \forall i, j \in I : \theta(\alpha_i) = -(\alpha_i + \alpha_j) &\implies a_{ij} \neq -1, \\ \forall i \in I : (\theta(\alpha_i))(h_i) &\neq -1. \end{aligned}$$

Satake diagrams can be defined as the following subset of compatible decorations of  $A$ :

$$(2.10) \quad \text{Sat}(A) = \{(X, \tau) \in \text{CDec}(A) \mid \forall i \in I \setminus X : i = \tau(i) \implies \zeta(\alpha_i) = 1\}.$$

It is well-known that Satake diagrams classify involutive Lie algebra automorphisms up to conjugacy, see e.g. [Ara62]. More precisely, in the current setup, for  $(X, \tau) \in \text{Sat}(A)$  and  $\gamma \in (\mathbb{C}^\times)^{I^*}$  define  $s_\gamma \in \check{H}$  by

means of

$$s_\gamma(\alpha_i) = \begin{cases} 1 & \text{if } i \in X, \\ \gamma_i & \text{if } i \in I^*, \\ \gamma_{\tau(i)}\zeta(\alpha_i) & \text{if } i \in (I \setminus X) \setminus I^*, \end{cases}$$

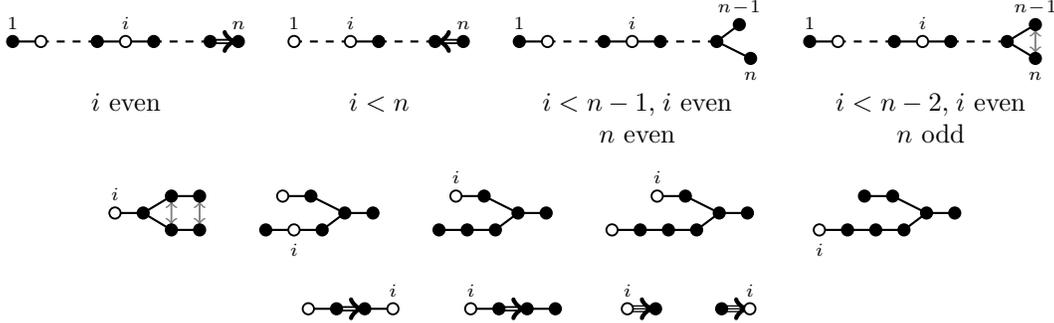
cf. [BK16, Eqs. (5.1-5.2)]. Then it follows from (2.8) that

$$(2.11) \quad \theta_\gamma := \text{Ad}(s_\gamma)\theta$$

satisfies  $(\theta_\gamma)^2 = \text{id}_{\mathfrak{g}}$ .

If  $(X, \tau) \in \text{CDec}(A) \setminus \text{GSat}(A)$  then there exists a pair  $(i, j) \in I \setminus X \times X$  such that the union of connected components of  $X$  neighbouring  $i$  is simply  $\{j\}$  and  $a_{ji} = -1$ . Hence  $\rho_X^\vee = \frac{1}{2}h_j$  so that  $\zeta(\alpha_i) = (-1)^{a_{ji}} = -1$  implying  $(X, \tau) \notin \text{Sat}(A)$ . Consequently  $\text{Sat}(A) \subseteq \text{GSat}(A)$ . The complement  $\text{GSat}(A) \setminus \text{Sat}(A)$  is empty if and only if  $A$  is of type  $A_n$ . We refer the reader to the classification in [He84, Table I], which does not explicitly distinguish between elements of  $\text{Sat}(A)$  and  $\text{GSat}(A) \setminus \text{Sat}(A)$ . It is convenient for our purposes to list the elements of  $\text{GSat}(A) \setminus \text{Sat}(A)$ , which we do in Table 1.

TABLE 1. All elements of  $\text{GSat}(A) \setminus \text{Sat}(A)$  for indecomposable Cartan matrices  $A$ . By a case-by-case analysis there is a unique  $i \in I$  such that  $\zeta(\alpha_i) = -1$ ; we have indicated the corresponding node in the diagrams. The classical diagrams are labelled in the usual way. For types  $C_n$  and  $D_n$  upper bounds on  $i$  are imposed to avoid the cases when  $\theta$  is an involution whose fixed-point subalgebra is isomorphic to  $\mathfrak{gl}_n$ .



Consider the real vector space  $V = \mathbb{R}\Phi$ . For a fixed  $\theta \in \text{Aut}^{\text{inv}}(\Phi)$  we can decompose  $V$  into the positive and negative  $\theta$ -eigenspaces,  $V = V^\theta \oplus V^{-\theta}$ . Denote by  $\bar{\cdot} : V \rightarrow V$  the corresponding projection onto  $V^{-\theta}$ . The *restricted roots* are the elements of

$$\bar{\Phi} = \{\bar{\alpha} \mid \alpha \in \Phi\} \setminus \{0\}.$$

Given an arbitrary  $\theta \in \text{Aut}^{\text{inv}}(\Phi)$ ,  $\bar{\Phi}$  is not necessarily a root system in its own right. According to [He84, Thm. 6.1],  $\bar{\Phi}$  is a (possibly non-reduced or empty) root system precisely if  $\theta = \theta(X, \tau) = -w_X\tau$ , where  $(X, \tau) \in \text{GSat}(A)$  or  $(X, \tau)$  is the diagram  $\circ \rightleftharpoons \bullet$ .

Now consider the following groups:

$$W^\theta = \{w \in W \mid w = \theta w \theta\} = \{w \in W \mid w = w_X \tau(w) w_X\},$$

$$\bar{W} = \{w|_{V^{-\theta}} \mid w \in W, w(V^{-\theta}) \subseteq V^{-\theta}\}.$$

If  $\theta = \theta(X, \tau)$  it follows straightforwardly that  $W_X$  is a subgroup of  $W^\theta$ . Moreover, [He84, Prop. 3.1] implies that  $\bar{W}$  is isomorphic to  $W^\theta/W_X$ . For  $i \in I^*$  we define  $\tilde{r}_i := w_X w_{X[i]} \in W$  where  $X[i] = X \cup \{i, \tau(i)\}$  and set  $s_i \in \text{GL}(V^{-\theta})$  to be the unique element satisfying  $s_i(\bar{\alpha}_i) = -\bar{\alpha}_i$  and  $s_i(\beta) = \beta$  for all  $\beta \in V^{-\theta}$  such that  $\beta(h_i) = 0$ . In [He84, Lemma 3.2, Thm. 3.3, Thm. 4.4] the following result is proved.

**Theorem 2.4.** *Let  $(X, \tau) \in \text{CDec}(A)$ . The following conditions are equivalent:*

- (i)  $(X, \tau) \in \text{GSat}(A)$ .
- (ii) For all  $i \in I^*$ ,  $s_i \in \overline{W}$ .
- (iii) For all  $i \in I^*$ ,  $\tilde{r}_i$  lies in  $W^\theta$  and satisfies  $\tilde{r}_i|_{V^{-\theta}} = s_i$ .
- (iv) For all  $i \in I^*$ ,  $\tau_{0, X[i]}$  preserves  $X$ .
- (v)  $\overline{W} = W(\overline{\Phi})$ .

In [Lu76, 5.9 (i)] it is shown that  $(\overline{W}, \{\tilde{r}_i\}_{i \in I^*})$  with  $\overline{W} = \langle \tilde{r}_i \rangle_{i \in I^*}$  is a Coxeter system if condition (iv) in Theorem 2.4 holds (also see [Lu02, 25.1]). If condition (iv) fails then for some  $i \in I^*$ ,  $w_{X[i]}$  and  $w_X$  do not commute so that  $\tilde{r}_i^2 \neq \text{id}_V$ . Hence we obtain the following result.

**Corollary 2.5.** *Let  $(X, \tau) \in \text{CDec}(A)$ . Then  $(\overline{W}, \{\tilde{r}_i\}_{i \in I^*})$  is a Coxeter system if and only if  $(X, \tau) \in \text{GSat}(A)$ .*

### 3. THE SUBALGEBRA $\mathfrak{k}$

For  $(X, \tau) \in \text{Sat}(A)$  and a suitable choice of  $\gamma \in (\mathbb{C}^\times)^{I^*}$  the  $\theta_\gamma$ -fixed subalgebra  $\mathfrak{k}$  of  $\mathfrak{g}$  can be presented in terms of generators, see e.g. [Ko14, Lemma 2.8]. This motivates the following seemingly *ad hoc* definition, where we permit a more general  $\gamma$ .

**Definition 3.1.** For  $(X, \tau) \in \text{CDec}(A)$  and  $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$  define  $\mathfrak{k}_\gamma = \mathfrak{k}_\gamma(X, \tau)$  to be the subalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{g}_X$ ,  $\mathfrak{h}^\theta$  and

$$(3.1) \quad b_{i, \gamma} = f_i + \gamma_i \theta(f_i) \quad \text{for all } i \in I \setminus X. \quad \emptyset$$

It is convenient to suppress the dependence on  $\gamma$  and simply write  $b_i$  and  $\mathfrak{k}$  if there is no cause for confusion. We denote  $b_i = f_i$  if  $i \in X$ . Since  $\mathfrak{h}_X \subseteq \mathfrak{h}^\theta$  it follows that  $\mathfrak{k}$  is generated by  $\mathfrak{n}_X^+ := \{e_i | i \in X\}$ ,  $\mathfrak{h}^\theta$  and  $b_i$  for  $i \in I$ . Owing to (2.1-2.2), these satisfy

$$(3.2) \quad [e_i, b_j] = \delta_{ij} h_i \in \mathfrak{h}^\theta \quad \text{for all } i \in X, j \in I,$$

$$(3.3) \quad [h, b_j] = -\alpha_j(h) b_j \quad \text{for all } h \in \mathfrak{h}^\theta, j \in I,$$

$$(3.4) \quad [h, e_j] = \alpha_j(h) e_j \quad \text{for all } h \in \mathfrak{h}^\theta, j \in X,$$

$$(3.5) \quad [h, h'] = 0 \quad \text{for all } h, h' \in \mathfrak{h}^\theta,$$

$$(3.6) \quad \text{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \quad \text{for all } i, j \in X, i \neq j.$$

By setting  $m = 1 - a_{ij}$  in Lemmas (A.1-A.3) one also obtains analogues of Serre relations among the generators  $b_i$ . Namely, for  $i, j \in I$  such that  $i \neq j$ ,

$$(3.7) \quad \text{ad}(b_i)^{1-a_{ij}}(b_j) = \begin{cases} (1 + \zeta(\alpha_i)) \gamma_i [\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j \in \Phi^-, a_{ij} = -1, \\ -18\gamma_i^2 e_j & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, a_{ij} = -3, \\ -\gamma_i (2h_i + h_j) & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, a_{ij} = -1, \\ (\gamma_i + \zeta(\alpha_i) \gamma_j) [\theta(f_i), f_j] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i) + \alpha_j \in \Phi^-, a_{ij} = 0, \\ \gamma_j h_i - \gamma_i h_j & \text{if } \theta(\alpha_i) + \alpha_j = 0, a_{ij} = 0, \\ 2(\gamma_i + \gamma_j) b_i & \text{if } \theta(\alpha_i) + \alpha_j = 0, a_{ij} = -1, \\ -\gamma_i b_j & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, a_{ij} = -1, \\ -3\gamma_i [b_i, b_j] & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, a_{ij} = -2, \\ -6\gamma_i^2 b_j - 3\gamma_i [b_i, [b_i, b_j]] & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, a_{ij} = -3, \\ 0 & \text{otherwise.} \end{cases}$$

In order to state the main result of this section, we need some more notation. Consider the subsets

$$I_{\text{diff}} = \{i \in I^* \mid i \neq \tau(i) \text{ and } (\theta(\alpha_i))(h_i) \neq 0\} = \{i \in I^* \mid i \neq \tau(i) \text{ and } \exists j \in X[i] \text{ s.t. } a_{ij} < 0\}$$

and

$$\Gamma = \Gamma(X, \tau) = \{\gamma \in (\mathbb{C}^\times)^{I \setminus X} \mid \forall i \in I^* : \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\text{diff}}\}.$$

For  $\mathbf{i} \in I^\ell$  with  $\ell \in \mathbb{Z}_{>0}$  we write  $\alpha_{\mathbf{i}} = \sum_{r=1}^\ell \alpha_{i_r}$  and

$$b_{\mathbf{i}} = \text{ad}(b_{i_1}) \cdots \text{ad}(b_{i_{\ell-1}})(b_{i_\ell}), \quad e_{\mathbf{i}} = \text{ad}(e_{i_1}) \cdots \text{ad}(e_{i_{\ell-1}})(e_{i_\ell}), \quad f_{\mathbf{i}} = \text{ad}(f_{i_1}) \cdots \text{ad}(f_{i_{\ell-1}})(f_{i_\ell}).$$

Observe that  $\mathfrak{n}^- = \text{Sp} \bigcup_{\ell > 0} \{f_{\mathbf{i}}\}_{\mathbf{i} \in I^\ell}$ . Hence for all  $\ell \in \mathbb{Z}_{>0}$  we can choose  $\mathcal{J}_\ell \subseteq I^\ell$  such that  $\{f_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{J}_\ell}$  is a basis for  $\text{Sp}\{f_{\mathbf{i}}\}_{\mathbf{i} \in I^\ell}$ . Then  $\{f_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{J}}$  with  $\mathcal{J} := \bigcup_{\ell \in \mathbb{Z}_{>0}} \mathcal{J}_\ell$  is a basis of  $\mathfrak{n}^-$ .

**Theorem 3.2.** *Let  $(X, \tau) \in \text{CDec}(A)$  and  $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$ . The following statements are equivalent:*

(i)  $(X, \tau) \in \text{GSat}(A)$  and  $\gamma \in \Gamma$ .

(ii) For all  $i, j \in I$  such that  $i \neq j$  we have the following bounded Serre relations:

$$(3.8) \quad \text{ad}(b_i)^{1-a_{ij}}(b_j) \in \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \oplus \bigoplus_{\substack{\mathbf{k} \in I^\ell \\ \alpha_{\mathbf{k}} < \lambda_{ij}}} \mathbb{C}b_{\mathbf{k}}$$

where  $\lambda_{ij} := (1 - a_{ij})\alpha_i + \alpha_j \in Q^+ \setminus \Phi^+$ .

(iii) We have the following identity for  $\mathfrak{h}^\theta$ -modules:

$$(3.9) \quad \mathfrak{k} = \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \oplus \bigoplus_{\mathbf{i} \in \mathcal{J}} \mathbb{C}b_{\mathbf{i}}.$$

(iv) We have

$$(3.10) \quad \mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta.$$

*Remark 3.3.* In the fixed-point case  $\mathfrak{k} = \mathfrak{g}^{\theta\gamma}$  (3.10) is trivially satisfied (note that  $\mathfrak{h}^\theta = \mathfrak{h}^{\theta\gamma}$ ). ∅

*Proof of Theorem 3.2.*

(i)  $\iff$  (ii): This is a direct consequence of (3.7).

(ii)  $\implies$  (iii): Owing to (3.3-3.5) it is sufficient to prove (3.9) as an identity for vector spaces. First we prove that  $\mathfrak{k} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \text{Sp}\{b_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{J}\}$ . From (3.2-3.3) it follows that, as vector spaces,

$$(3.11) \quad \mathfrak{k} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \langle b_j \rangle_{j \in I} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \sum_{\ell \in \mathbb{Z}_{>0}} \sum_{\mathbf{i} \in I^\ell} \mathbb{C}b_{\mathbf{i}}.$$

As a consequence of this, we see that it suffices to prove that for all  $\mathbf{j} \in \bigcup_{\ell} I^\ell$  we have

$$(3.12) \quad b_{\mathbf{j}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \text{Sp}\{b_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{J}\}.$$

We will prove this by induction with respect to the height  $\ell$ . Since for all  $j \in I$  we have  $\dim(\mathfrak{g}_{-\alpha_j}) = 1$  and hence  $(j) \in \mathcal{J}$ , the case  $\ell = 1$  is trivial. Now fix  $\ell \in \mathbb{Z}_{>1}$  and assume that (3.12) holds true for all smaller positive integers. Fix  $\mathbf{j} \in I^\ell$  and repeatedly apply the Serre relations (2.2) to obtain that for all  $\mathbf{i} \in \mathcal{J}_\ell$  there exist  $a_{\mathbf{i}} \in \mathbb{C}$  such that

$$f_{\mathbf{j}} = \sum_{\mathbf{i} \in \mathcal{J}_\ell} a_{\mathbf{i}} f_{\mathbf{i}}.$$

Hence, by virtue of (ii) and equations (3.2-3.3) it follows that

$$b_{\mathbf{j}} - \sum_{\mathbf{i} \in \mathcal{J}_\ell} a_{\mathbf{i}} b_{\mathbf{i}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \text{Sp}\left\{b_{\mathbf{i}} \mid \mathbf{i} \in \bigcup_{m=1}^{\ell-1} I^m\right\}.$$

Using the induction hypothesis for the elements  $b_{\mathbf{i}}$  in the last summation one obtains (3.12).

It remains to show that the sum in (3.12) is direct. Let  $\mathbf{j} \in \mathcal{J}$ . Then  $f_{\mathbf{j}}$  is nonzero. Because of the explicit formula (3.1) we have

$$(3.13) \quad b_{\mathbf{j}} - f_{\mathbf{j}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \mathbb{C}\theta(f_{\mathbf{j}}) + \text{Sp}\{b_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{J}, \alpha_{\mathbf{i}} < \alpha_{\mathbf{j}}\}.$$

Hence  $f_j = \pi_{-\alpha_j}(b_j)$  for all  $j \in \mathcal{J}$ , where  $\pi_\alpha$  is the projection on  $\mathfrak{g}_\alpha$  for  $\alpha \in \Phi$ , see (2.3). Thus the linear independence of  $\{f_j\}_{j \in \mathcal{J}}$  together with (2.3) implies that the sum is direct.

- (iii)  $\implies$  (iv): By definition,  $\mathfrak{h}^\theta \subseteq \mathfrak{k} \cap \mathfrak{h}$  so it suffices to show that  $\mathfrak{k} \cap \mathfrak{h} \subseteq \mathfrak{h}^\theta$ . Suppose  $h \in \mathfrak{k} \cap \mathfrak{h}$ . Since  $\pi_{-\alpha_j}(b_j) = f_j$  and the triangular decomposition (2.3), part (iii) implies  $h \in \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \subseteq \mathfrak{g}^\theta$  so  $h \in \mathfrak{h}^\theta$ .
- (iv)  $\implies$  (ii): We prove the contrapositive. If (3.8) fails then (3.14) and (3.7) imply that either  $\gamma_j h_i - \gamma_i h_j \in \mathfrak{k} \cap (\mathfrak{h} \setminus \mathfrak{h}^\theta)$  with  $\gamma_i \neq \gamma_j$  or  $2h_i + h_j \in \mathfrak{k} \cap (\mathfrak{h} \setminus \mathfrak{h}^\theta)$ . In either case (3.10) fails.  $\square$

It is convenient to have an explicit description of  $\mathfrak{h}^\theta$ . Given  $i \in I$ , by applying  $\theta$  to  $\theta(h_i) - h_i - \theta(h_{\tau(i)}) + h_{\tau(i)} \in \mathfrak{g}_X \cap \mathfrak{h}$  one obtains  $\theta(h_i - h_{\tau(i)}) = h_i - h_{\tau(i)}$ . From this we straightforwardly deduce

$$(3.14) \quad \mathfrak{h}^\theta = \bigoplus_{i \in X} \mathbb{C}h_i \oplus \bigoplus_{\substack{i \in I^* \\ i \neq \tau(i)}} \mathbb{C}(h_i - h_{\tau(i)})$$

We denote  $\Phi_X = \Phi \cap Q_X$  and note that  $|\mathcal{J}| = |\Phi|/2$ ; from (3.14) we also obtain  $\dim(\mathfrak{h}^\theta) = |I| - |I^*|$ . Hence, given  $(X, \tau) \in \text{GSat}(A)$  and  $\gamma \in \Gamma$ , Theorem 3.2 (iii) implies

$$(3.15) \quad \dim(\mathfrak{k}) = |\Phi_X|/2 + |I| - |I^*| + |\Phi|/2.$$

**Corollary 3.4.** *Let  $(X, \tau) \in \text{GSat}(A)$  and  $\gamma \in \Gamma$ . The generating set*

$$\{h_i, e_i\}_{i \in X} \cup \{h_i - h_{\tau(i)}\}_{i \in I^*, i \neq \tau(i)} \cup \{b_i\}_{i \in I},$$

and the relations (3.2-3.6) provide a presentation of  $\mathfrak{k}$ .

*Proof.* There are no relations for the  $b_i$  other than (3.2), (3.3) and (3.7): otherwise applying  $\pi_{-\alpha}$  with  $\alpha \in \Phi^+$  maximal produces a relation for the  $f_i$  inequivalent to a relation (2.1), (2.2).  $\square$

**3.1. Ideal structure of  $\mathfrak{k}$ .** In this section we assume that  $A$  is indecomposable, so that  $\mathfrak{g}$  is simple. In order to describe the derived subalgebra of  $\mathfrak{k}$  recall the set  $I_{\text{diff}} \in I^*$  and define

$$(3.16) \quad \begin{aligned} I_{\text{ns}} &= \{i \in I \mid (\theta(\alpha_i))(h_i) = -2\} = \{i \in I \mid i = \tau(i), \check{X}(i) = \emptyset\}, \\ I_{\text{nsf}} &= \{j \in I_{\text{ns}} \mid \forall i \in I_{\text{ns}} a_{ij} \in 2\mathbb{Z}\}. \end{aligned}$$

**Proposition 3.5.** *Let  $(X, \tau) \in \text{GSat}(A)$  and  $\gamma \in \Gamma$ . As vector spaces we have*

$$\mathfrak{k} = \mathfrak{k}' \oplus \bigoplus_{i \in I_{\text{diff}}} \mathbb{C}(h_i - h_{\tau(i)}) \oplus \bigoplus_{i \in I_{\text{nsf}}} \mathbb{C}b_i.$$

*Proof.* Fix  $(X, \tau) \in \text{GSat}(A)$ . Note that neither  $h_i - h_{\tau(i)}$  ( $i \in I_{\text{diff}}$ ) nor  $b_j$  ( $j \in I_{\text{nsf}}$ ) is a linear combination of Lie brackets in  $\mathfrak{k}$ . This follows from Corollary 3.4 and (3.2-3.7): these elements do not appear as in the expressions for Lie brackets in the defining relations of  $\mathfrak{k}$ .

It now suffices to show that the remaining basis elements specified in (3.9) are linear combinations of Lie brackets in  $\mathfrak{k}$ , for which we argue as follows.

$b_i$  with  $i \in \mathcal{J}_\ell$ ,  $\ell > 1$ : This holds by definition.

$e_i, f_i, h_i$  with  $i \in X$ : This follows from (3.2-3.4).

$h_i - h_{\tau(i)}$  with  $i \in I^* \setminus I_{\text{diff}}$  and  $i \neq \tau(i)$ : The given condition is equivalent to  $w_X(\alpha_i) = \alpha_i$  and  $a_{i\tau(i)} = 0$ .

Hence (3.7) implies that  $h_i - h_{\tau(i)} = \gamma_i^{-1}[b_i, b_{\tau(i)}]$ .

$b_j$  with  $\check{X}(j) \neq \emptyset$ : There exists  $i \in X$  such that  $a_{ij} \neq 0$ . By (3.3) we have  $b_j = -a_{ij}^{-1}[h_i, b_j]$ .

$b_j$  with  $j \neq \tau(j)$ : Note that  $a_{\tau(j)j} \leq 0$ . By (3.3) we have  $b_j = (a_{\tau(j)j} - 2)^{-1}[h_j - h_{\tau(j)}, b_j]$ .

$b_j$  with  $j \in I_{\text{ns}} \setminus I_{\text{nsf}}$ : By definition of  $I_{\text{nsf}}$  there exists  $i \in I_{\text{ns}}$  such that  $a_{ij} \in \{-1, -3\}$ . According to (3.7),  $b_j = -\gamma_i^{-1} \text{ad}(b_i)^2(b_j)$  if  $a_{ij} = -1$  and  $b_j = -(2\gamma_i)^{-1} \text{ad}(b_i)^2(b_j) - (6\gamma_i^2)^{-1} \text{ad}(b_i)^4(b_j)$  if  $a_{ij} = -3$ ; in either case  $b_j \in \mathfrak{k}'$ .  $\square$

It follows that the codimension of  $\mathfrak{k}'$  in  $\mathfrak{k}$  equals  $|I_{\text{diff}}| + |I_{\text{nsf}}|$ . For  $(X, \tau) \in \text{Sat}(A)$ , in [Let02, Sec. 7, Variation 1] it was noted that  $|I_{\text{diff}}| \leq 1$  if  $A$  is of finite type. In light of the above it is natural to generalize this in two directions: also involve the set  $I_{\text{nsf}}$  and allow  $(X, \tau) \in \text{GSat}(A)$ . It turns out the same upper bound holds true and there are generalized Satake diagrams with  $|I_{\text{diff}}| + |I_{\text{nsf}}| = 1$  unless  $A$  is of type  $E_8$ ,  $F_4$  or  $G_2$ . From Table 1 it follows that the only elements of  $\text{GSat}(A) \setminus \text{Sat}(A)$  for which  $|I_{\text{diff}}| + |I_{\text{nsf}}| = 1$  are of the form  $\overset{1}{\circ} - \overset{2}{\circ} - \cdots - \overset{n}{\bullet} \leftarrow \bullet$  with  $n > 2$  in which case  $I_{\text{nsf}} = \{1\}$  and  $\zeta(\alpha_2) = -1$ .

For the reasons that will become clear a bit later we introduce a further refinement of generalized Satake diagrams. In particular, we define the set of *weak Satake diagrams* by

$$\text{WSat}(A) = \{(X, \tau) \in \text{GSat}(A) \setminus \text{Sat}(A) \mid (X, \tau) \text{ contains no minimal subdiagram of the form } \bullet \Rightarrow \circ\}.$$

As mentioned in Table 1, for elements of  $\text{GSat}(A) \setminus \text{Sat}(A)$  a case-by-case analysis yields that there can be at most one  $i \in I \setminus X$  such that  $i = \tau(i)$  and  $\zeta(\alpha_i) = -1$ . For  $(X, \tau) \in \text{WSat}(A)$  we will obtain a semidirect product decomposition in terms of a reductive Lie subalgebra and a nilpotent ideal in which this unique  $i \in I \setminus X$  plays an important role.

For any  $r \in \mathbb{Z}_{\geq 0}$  and any  $i \in I$  denote by  $\mathfrak{k}(i)_r$  the span of all  $b_j$  such that the coefficient of  $\alpha_i$  in  $\alpha_j$  is precisely  $r$ . We then have the following decomposition

$$\langle b_i \rangle_{i \in I} = \bigoplus_{r=0}^{\infty} \mathfrak{k}(i)_r.$$

Consider the subspace

$$\mathfrak{k}(i) := \bigoplus_{r=1}^{\infty} \mathfrak{k}(i)_r$$

and the subalgebras

$$\mathfrak{k}_i := \langle \mathfrak{n}_X^+, \mathfrak{h}^\theta, \{b_j\}_{j \in I \setminus \{i\}} \rangle \subseteq \mathfrak{k}, \quad \mathfrak{g}_i := \langle \{e_j, f_j, h_j\}_{j \in I \setminus \{i\}} \rangle \subset \mathfrak{g}.$$

Note that  $\mathfrak{k} = \mathfrak{k}_i + \mathfrak{k}(i)$  (not necessarily a direct sum, since e.g.  $b_i$  may lie in  $\mathfrak{k}_i$ ).

**Proposition 3.6.** *Let  $(X, \tau) \in \text{WSat}(A)$  and  $\gamma \in \Gamma$ . Denote by  $i$  the unique element of  $I \setminus X$  such that  $i = \tau(i)$  and  $\zeta(\alpha_i) = -1$ . Then  $\mathfrak{k}(i)_r = \{0\}$  if  $r > 2$  and we have the lower central series*

$$\mathfrak{k}(i) = \mathfrak{k}(i)_1 \oplus \mathfrak{k}(i)_2 \supset \mathfrak{k}(i)_2 \supset \{0\}$$

so that  $\mathfrak{k}(i)$  is nilpotent of class 2. Furthermore, both  $\mathfrak{k}(i)_1$  and  $\mathfrak{k}(i)_2$  are  $\mathfrak{k}_i$ -modules under the adjoint action,  $\mathfrak{k}(i)$  is an ideal of  $\mathfrak{k}$ ,  $\mathfrak{k}_i$  is the fixed-point subalgebra of  $\theta|_{\mathfrak{g}_i}$  and we have  $\mathfrak{k} = \mathfrak{k}_i \ltimes \mathfrak{k}(i)$ .

*Proof.* Note that (3.7) implies, for all  $j \in I \setminus \{i\}$ , that

$$(3.17) \quad \text{ad}(b_i)^{1-a_{ij}}(b_j) = 0$$

$$(3.18) \quad \text{ad}(b_j)^{1-a_{ji}}(b_i) \in \sum_{r=1}^{-a_{ij}} \mathbb{F} \text{ad}(b_j)^r(b_i) \subseteq \mathfrak{k}(i)_1.$$

Since (3.3) and (3.18) are the only relations in  $\mathfrak{k}$  with  $b_i$  appearing on the right-hand side, it follows that  $\mathfrak{k}_i = \langle \mathfrak{n}_X^+, \mathfrak{h}^\theta, \mathfrak{k}(i)_0 \rangle$  and  $\mathfrak{k} = \mathfrak{k}_i \oplus \mathfrak{k}(i)$  (as vector spaces). Deleting the node  $i$  from any diagram in Table 1 one obtains a (possibly disconnected) Satake diagram such that  $\theta|_{\mathfrak{g}_i}$  by virtue of (2.8) is an involution. From Table 1 it also follows that  $I^* = I \setminus X$  so that  $\mathfrak{k}_i$  is the fixed-point subalgebra of  $\mathfrak{g}_i$  for the involution  $\theta_\gamma$ , see (2.11).

Combined with (3.2-3.3), (3.18) implies that each summand  $\mathfrak{k}(i)_r$  is a  $\mathfrak{k}_i$ -module. Hence  $\mathfrak{k}(i)$  is a  $\mathfrak{k}_i$ -module and by virtue of (3.17) it is a subalgebra of  $\mathfrak{k}$ . It follows that  $\mathfrak{k}(i)$  is an ideal. Automatically we have that  $[\oplus_{r=1}^s \mathfrak{k}(i)_r, \mathfrak{k}(i)_1] \subseteq \oplus_{r=1}^{s+1} \mathfrak{k}(i)_r$  for all  $s \in \mathbb{Z}_{\geq 1}$ . A case-by-case analysis using Table 1 yields that the coefficient in front of  $\alpha_i$  in the highest root of  $\Phi$  is always 2. This implies  $\mathfrak{k}(i)_3 = 0$  so that  $\mathfrak{k}(i)_2$  is the centre of  $\mathfrak{k}(i)$  and we obtain the indicated lower central series.  $\square$

Regarding the centre  $\mathfrak{z}$  of  $\mathfrak{k}$  for  $(X, \tau) \in \text{WSat}(A)$ , recall the notation  $i$  for the unique element of  $I \setminus X$  such that  $i = \tau(i)$  and  $\zeta(\alpha_i) = -1$ . Since the centre of the ideal  $\mathfrak{k}(i)$  is  $\mathfrak{k}(i)_2$ , we must have  $\mathfrak{z} \subseteq \mathfrak{k}(i)_2$ . Define

$$\mathcal{J}_{\text{even}} := \{j \in \mathcal{J} \mid \forall k \in I \setminus X \text{ the coefficient of } \alpha_j \text{ in front of } \alpha_k \text{ is even}\}$$

so that

$$\mathfrak{k}(i)_{2,\text{even}} := \bigoplus_{j \in \mathcal{J}_{\text{even}}} \mathbb{C}b_j \subset \mathfrak{k}(i)_2.$$

We claim without proof that  $\mathfrak{z}$  is generated by a single element of  $\mathfrak{k}(i)_{2,\text{even}}$ .

Let us now explain the motivation behind the definition of the set  $\text{WSat}(A)$ . Consider the excluded generalized Satake diagram  $\bullet \rightarrow \circ$ . By definition,  $\mathfrak{k}$  is the subalgebra of  $\mathfrak{g} = \text{Lie}(G_2)$  generated by  $e_1, h_1, b_1 = f_1$  and  $b_2 = f_2 + \gamma_2 \theta(f_2)$  for some  $\gamma_2 \in \mathbb{C}^\times$ . The relations (3.2-3.7) specialize to

$$\begin{aligned} [e_1, b_1] &= h_1, & [e_1, b_2] &= 0, & [h_1, b_1] &= -2b_1, & [h_1, b_2] &= b_2, & [h_1, e_1] &= 2e_1, \\ [b_1, [b_1, b_2]] &= 0, & [b_2, [b_2, [b_2, b_1]]] &= -18\gamma_2^2 e_1. \end{aligned}$$

According to (3.15) we have  $\dim(\mathfrak{k}) = 8$ . A natural basis is given by

$$e_1, \quad b_1, \quad h_1, \quad b_2, \quad b_{(2,1)}, \quad b_{(2,2,1)}, \quad b_{(2,2,2,1)}, \quad b_{(1,2,2,2,1)}.$$

Using the adjoint action of  $e_1, b_1$  and  $b_2$  on  $\mathfrak{k}$  it is easy to verify that an ideal of  $\mathfrak{k}$  equals  $\mathfrak{k}$  if it contains any of the generators listed above. This together with some straightforward computations shows that  $\mathfrak{k}$  is in fact a simple Lie algebra. Since  $\dim(\mathfrak{k}) = 8$ , it must be isomorphic to  $\mathfrak{sl}_3$ . On the other hand, if  $(X, \tau) \in \text{WSat}(A)$ , since  $\mathfrak{k}$  has a nonzero nilpotent ideal by Proposition 3.6,  $\mathfrak{k}$  is not a reductive Lie algebra.

**Proposition 3.7.** *Let  $(X, \tau) \in \text{GSat}(A) \setminus \text{Sat}(A)$  and  $\gamma \in \Gamma$ . Then  $\mathfrak{k}$  is not the fixed-point subalgebra of any automorphism of  $\mathfrak{g}$ .*

*Proof.* We first show this for the case when  $(X, \tau)$  is  $\bullet \rightarrow \circ$ . Suppose there exists  $\phi \in \text{Aut}(\mathfrak{g})$  such that  $\mathfrak{k} = \mathfrak{g}^\phi$ . From  $[h_2, b_1] = 3b_1$  and  $[h_2, e_1] = -3e_1$  one establishes straightforwardly that  $\phi(h_2) \in \mathfrak{h}$  and hence that  $\phi(h_2) = \frac{3}{2}(m-1)h_1 + mh_2$  for some  $m \in \mathbb{C}$ . Next, from  $\theta(f_2) = e_{(2,1)}$  it follows that  $[h_2, b_2] = -f_2 - b_2$ ; hence  $\phi(f_2) = mf_2 + \frac{1}{2}(1-m)b_2$ . Combining this with  $[f_2, b_2] = 3e_1$  one obtains  $m = 1$ . But this means that  $h_2$  and  $f_2$  are also fixed points of  $\phi$ , contrary to assumption. Hence such  $\phi$  does not exist. Now let  $(X, \tau) \in \text{WSat}(A)$ . In this case  $\mathfrak{k}$  is not a reductive Lie algebra and [Ja62, Thm. 1] implies that  $\mathfrak{k}$  cannot be the fixed-point subalgebra of any automorphism of  $\mathfrak{g}$ .  $\square$

Nevertheless, in Section 4 we will show that for all  $(X, \tau) \in \text{GSat}(A)$ , the subalgebra  $\mathfrak{k}$  can be quantized resulting in a coideal subalgebra possessing a universal  $K$ -matrix.

**3.2. The universal enveloping algebra  $U(\mathfrak{k})$ .** Let  $(X, \tau) \in \text{GSat}(A)$  and  $\gamma \in \Gamma$ . We identify  $\mathfrak{k}$  with its image in  $U(\mathfrak{k})$  under the canonical Lie algebra embedding. The generators of  $U(\mathfrak{k})$  corresponding to  $b_i$  ( $i \in I \setminus X$ ) can be modified by scalar terms, which is a straightforward generalization of [Ko14, Cor. 2.9].

**Proposition 3.8.** *For  $(X, \tau) \in \text{GSat}(A)$ ,  $\gamma \in \Gamma$  and  $\mathbf{s} \in \mathbb{C}^{I \setminus X}$ , the universal enveloping algebra  $U(\mathfrak{k}_\gamma)_\mathbf{s}$  is generated by  $e_i, f_i$  ( $i \in X$ ),  $h \in \mathfrak{h}^\theta$  and*

$$(3.19) \quad b_{i,\gamma,\mathbf{s}} = f_i + \gamma_i \theta(f_i) + s_i \quad \text{for all } i \in I \setminus X.$$

Again, if there is no cause for confusion, we will suppress  $\gamma$  and  $\mathbf{s}$  from the notation. Because of Corollary 3.4 we immediately obtain the following result, which addresses [Ko14, Rmk. 2.10].

**Proposition 3.9.** *For  $(X, \tau) \in \text{GSat}(A)$ ,  $\gamma \in \Gamma$  and  $\mathbf{s} \in \mathbb{C}^{I \setminus X}$ , the defining relations of the universal enveloping algebra  $U(\mathfrak{k})$  are given by (3.2-3.6), with the Lie bracket interpreted as commutator.*

We may view  $U(\mathfrak{k})$  as a Hopf subalgebra of  $U(\mathfrak{g})$  so that Lie algebra automorphisms of  $\mathfrak{g}$  lift to Hopf algebra automorphisms of  $U(\mathfrak{g})$ . Call two Hopf subalgebras  $B, B'$  of  $U(\mathfrak{g})$  equivalent if there exists  $\phi \in \text{Aut}_{\text{Hopf}}(U(\mathfrak{g}))$  such that  $B' = \phi(B)$ . Define

$$(3.20) \quad \begin{aligned} \tilde{\Gamma} &:= \{\gamma \in \Gamma \mid \gamma_i = 1 \text{ unless } i \in I_{\text{diff}}\}, \\ \mathcal{S} &:= \{\mathbf{s} \in \mathbb{C}^{I \setminus X} \mid s_i = 0 \text{ unless } i \in I_{\text{nsf}}\}. \end{aligned}$$

**Proposition 3.10.** *Let  $(X, \tau) \in \text{GSat}(A)$ ,  $\gamma \in \Gamma$  and  $\mathbf{s} \in \mathbb{C}^{I \setminus X}$ . There exist  $\tilde{\gamma} \in \tilde{\Gamma}$  and  $\mathbf{s}' \in \mathcal{S}$  such that  $U(\mathfrak{k}_{\gamma})_{\mathbf{s}}$  is equivalent to  $U(\mathfrak{k}_{\tilde{\gamma}})_{\mathbf{s}'}$ .*

*Proof.* The existence of  $\tilde{\gamma}$  can be proven in an argument entirely analogous to the proof of [Ko14, Prop. 9.2 (i)]. It follows that  $U(\mathfrak{k}_{\gamma})_{\mathbf{s}}$  is equivalent to  $U(\mathfrak{k}_{\tilde{\gamma}})_{\tilde{\mathbf{s}}}$  for some  $\tilde{\mathbf{s}} \in \mathbb{C}^{I \setminus X}$ .

Regarding the existence of  $\mathbf{s}' \in \mathcal{S}$ , note that  $b_{i, \tilde{\gamma}} \in (\mathfrak{k}_{\tilde{\gamma}})'$  unless  $i \in I_{\text{nsf}}$  owing to Prop. 3.5. Hence  $U(\mathfrak{k}_{\tilde{\gamma}})_{\tilde{\mathbf{s}}}$  is already generated by  $e_i, f_i$  ( $i \in X$ ),  $h \in \mathfrak{h}^{\theta}$ ,  $b_{i, \tilde{\gamma}, 0}$  for  $i \in (I \setminus X) \setminus I_{\text{nsf}}$  and  $b_{i, \tilde{\gamma}, \tilde{\mathbf{s}}}$  for  $i \in I_{\text{nsf}}$ . Hence we may take  $s'_i = \tilde{s}_i$  if  $i \in I_{\text{nsf}}$  and  $s'_i = 0$  otherwise.  $\square$

#### 4. THE UNIVERSAL $K$ -MATRIX REVISITED

Assume the  $d_i$  are dyadic rationals and let  $\mathbb{K}$  be a quadratic closure of  $\mathbb{C}(q)$  where  $q$  is an indeterminate, so that  $q_i := q^{d_i} \in \mathbb{K}$  for all  $i \in I$ . The Drinfeld-Jimbo quantum group  $U_q = U_q(\mathfrak{g})$  is an associative unital algebra over  $\mathbb{K}$  which quantizes the universal enveloping algebra  $U(\mathfrak{g})$ . It is generated by  $\{E_i, F_i, t_i^{\pm 1}\}$  where  $i \in I$ , satisfying the relations given in e.g. [Lu94, 3.1.1]. It is a Hopf algebra whose structure is defined by the choice of the coproduct:

$$\Delta(E_i) = E_i \otimes 1 + t_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes t_i^{-1} + 1 \otimes F_i, \quad \Delta(t_i) = t_i \otimes t_i.$$

For  $\alpha = \sum_i n_i \alpha_i \in Q$  with  $n_i \in \mathbb{Z}$  we write  $t_{\alpha} = \prod_{i \in I} t_i^{n_i}$ . The Hopf subalgebra  $U_q^0 = U_q(\mathfrak{h})$  is the subalgebra generated by  $t_i^{\pm 1}$  for  $i \in I$  and spanned by  $\{t_{\alpha}\}_{\alpha \in Q}$ . In terms of the quantum root spaces

$$(U_q)_{\alpha} = \{u \in U_q \mid \forall i \in I \ t_i u t_i^{-1} = q_i^{\alpha(h_i)} u\}$$

where  $\alpha \in Q$ , we have the  $Q$ -grading

$$(4.1) \quad U_q = \bigoplus_{\alpha \in Q} (U_q)_{\alpha}, \quad (U_q)_{\alpha} (U_q)_{\beta} \subseteq (U_q)_{\alpha + \beta}.$$

According to [Tw92, Thm. 2.1] we have  $\text{Aut}_{\text{Hopf}}(U_q) = \text{Ad}(\tilde{H}) \rtimes \text{Aut}(A)$  with  $\text{Ad}(\chi)$  for  $\chi \in \tilde{H}$  acting on the root space  $(U_q)_{\alpha}$  for  $\alpha \in Q$  by multiplication by  $\chi(\alpha)$ , and  $\text{Aut}(A)$  acting by relabelling. Other relevant algebra automorphisms are Lusztig's automorphisms  $T_i$  for  $i \in I$  given as  $T''_{i,1}$  in [Lu94, 37.1.3] which define a braid group action on  $U_q$  restricting to the Weyl group action on  $U_q^0$ :  $T_i(t_{\alpha}) = t_{r_i(\alpha)}$  for  $i \in I$  and  $\alpha \in Q$ . For  $X \subseteq I$  with  $w_X = r_{i_1} \cdots r_{i_\ell}$  a reduced decomposition we define  $T_X = T_{i_1} \cdots T_{i_\ell}$ . Also, we define a quantum analogue of the Chevalley involution by

$$(4.2) \quad \omega_q(E_i) = -t_i^{-1} F_i, \quad \omega_q(F_i) = -E_i t_i, \quad \omega_q(t_i^{\pm 1}) = t_i^{\mp 1}$$

for  $i \in I$ . Then  $\omega_q$  commutes with  $\text{Aut}(A)$  and with  $T_i$  for  $i \in I$ , see [BK16, Lemma 7.1]. Assuming  $\tau(X) = X$ , one straightforwardly checks that  $\tau$  commutes with  $T_X$ .

**4.1. Quantum pair algebras.** We will follow the approach of the papers [Ko14, BK15, BK16] and simply highlight where a definition or formula needs to be changed. The quantum analogon of the map  $\theta = \text{Ad}(w_X) \tau \omega$  is the map

$$\theta_q = \theta_q(X, \tau) = T_X \tau \omega_q \in \text{Aut}_{\text{alg}}(U_q).$$

Note the absence of the factor  $\text{Ad}(s)$ , cf. [Ko14, Def. 4.3] or [BK16, Def. 5.4 and Eqn. (5.4)], which was present in *ibid.* to guarantee that  $\theta_q$  specializes to the appropriate Lie algebra involution, see [Ko14, Prop. 10.2]. Similar to (3.14) it follows that  $U_q(\mathfrak{h})^{\theta_q}$  consists of polynomials in  $t_i^{\pm 1}$  ( $i \in X$ ) and  $(t_i t_{\tau(i)}^{-1})^{\pm 1}$  ( $i \in I^*, i \neq \tau(i)$ ).

It is equal to the subalgebra denoted  $U_{\mathcal{O}}'$  in [Ko14].

The quantization of the fixed-point subalgebra in the formalism by [Ko14] relies on the presentation of  $\mathfrak{g}^{\theta\gamma}$  in terms of generators given in [Ko14, Lemma 2.8]. Our  $\mathfrak{k}(X, \tau)$  with  $(X, \tau) \in \text{GSat}(A)$  by definition can be quantized to a right coideal subalgebra in the same way.

**Definition 4.1.** Let  $(X, \tau) \in \text{GSat}(A)$ ,  $\gamma \in (\mathbb{K}^\times)^{I \setminus X}$  and  $\mathbf{s} \in \mathbb{K}^{I \setminus X}$ . Then  $B = B_{\gamma, \mathbf{s}}(X, \tau)$  is the coideal subalgebra generated by  $U_q(\mathfrak{g}_X)$ ,  $U_q(\mathfrak{h})^{\theta_q}$  and the elements

$$B_i = B_{i; \gamma, \mathbf{s}} = F_i + \gamma_i \theta_q(F_i t_i) t_i^{-1} + s_i t_i^{-1} \quad \text{for all } i \in I \setminus X. \quad \varnothing$$

To make a direct match between the Kolb-Balogović formalism based on fixed-point subalgebras and our more general approach one should set, for all  $i \in I \setminus X$ ,

$$\gamma_i = s(\alpha_{\tau(i)}) c_i,$$

see also [BK16, Eqn. (7.7)]. If the tuples  $\gamma, \mathbf{s}$  lie in the sets

$$(4.3) \quad \begin{aligned} \Gamma_q &= \{\gamma \in (\mathbb{K}^\times)^{I \setminus X} \mid \forall i \in I^* \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\text{diff}}\}, \\ \mathcal{S}_q &= \{\mathbf{s} \in \mathbb{K}^{I \setminus X} \mid s_i = 0 \text{ unless } i \in I_{\text{nsf}}\} \end{aligned}$$

respectively, then according to [Ko14, Sec. 5.3 and Sec. 6] one obtains decompositions of  $B$  yielding the quantum analogue of (3.10), namely  $B \cap U_q(\mathfrak{h}) = U_q(\mathfrak{h})^{\theta_q}$ . The key condition for Satake diagrams, see (2.10), is only used in [Ko14, Proof of Lemma 5.11, Step 1], but it is clear that what is needed is precisely the weaker condition appearing in the definition of a generalized Satake diagram, see Definition 2.2. The rest of [Ko14] is applicable without change in the setting of generalized Satake diagrams; in particular in the specialization ( $q \rightarrow 1$ ) one recovers  $U(\mathfrak{k})$ , see [Ko14, Sec. 10].

In [BK15] the bar involutions for  $U_q$  and  $B$  are studied, following earlier work by [ES13] and [BW13] in the case of quantum symmetric pairs of  $\mathfrak{gl}_N$  type. The proof of [BK15, Prop. 2.3] relies on a case-by-case analysis of Satake diagrams of finite type from Araki's work [Ara62]. We claim here without proof that a similar analysis using Table 1 yields the same result for all generalized Satake diagrams, in other words that [BK15, Prop. 2.5] holds with  $\nu_i = 1$  for all  $i \in I \setminus X$  (otherwise  $\nu_i = -1$ ). In the remainder of [BK15] the defining condition of Satake diagrams or a case-by-case analysis is not used so that these results remain valid.

The universal  $K$ -matrix for the algebra  $B$  is constructed in [BK16] in the case  $(X, \tau) \in \text{Sat}(A)$ . We restate some key conditions in terms of the parameters  $\gamma$ . Assuming  $\nu_i = 1$  for all  $i \in I \setminus X$ , condition [BK16, Eqn. (5.17)] is equivalent to

$$\gamma_{\tau(i)} = \zeta(\alpha_i) q_i^{\theta(\alpha_i) - 2\rho_X(h_i)} \overline{\gamma_i},$$

where  $\rho_X$  is the Weyl vector of  $\mathfrak{g}_X$  and  $\bar{\cdot}$  denotes the bar involution of  $U_q$ , which by definition fixes  $E_i, F_i$  and inverts  $t_i^{\pm 1}$  and  $q$ . In [BK16, Proof of Lemma 6.4] the defining condition of Satake diagrams is used, but as before the defining condition of generalized Satake diagrams is what is needed. Then [BK16, Eqn. (7.14)] needs to be replaced by

$$\overline{T_{w_X}(E_{\tau(i)})} = \zeta(\alpha_i) q_i^{-2\rho_X(h_i)} T_{w_X}^{-1}(E_{\tau(i)})$$

so that the scalar  $\rho_i$  appearing in [BK16, Lemma 9.3] equals  $q_i^{-\theta(\alpha_i)(h_i)} \gamma_{\tau(i)}$  since [BK16, Eqn. (9.8)] is equivalent to

$$\overline{\gamma_i T_{w_X}(E_{\tau(i)})} = q_i^{-\theta(\alpha_i)(h_i)} \gamma_{\tau(i)} T_{w_X}^{-1}(E_{\tau(i)}).$$

Finally, we highlight the paper [DK18] which establishes an elegant factorization property of the quasi  $K$ -matrix in terms of the restricted Weyl group of  $\mathfrak{g}$ . Sections 2.2 and 2.3 in *ibid.* entail an analysis of the restricted Weyl group and restricted root system following [Lu76]. For completeness, in reference to a comment in [DK18, between Eqs. (2.9) and (2.10)] we remark that also for all  $(X, \tau) \in \text{GSat}(A) \setminus \text{Sat}(A)$  the set  $X$  is invariant under the diagram automorphism  $\tau_0 = \tau_{I,0}$  corresponding to the longest element of  $W$ ; this follows from Table 1. The upshot of this in [DK18] is that  $\tau_{0, X[i]}$  stabilizes  $X$  (for all  $i \in I^*$ ). This is used to derive that the  $\tilde{\tau}_i = w_X w_{X[i]}$  form a Coxeter system for the group they generate. Alternatively, this result follows from Corollary 2.5 for all generalized Satake diagrams.

A. DERIVING MODIFIED SERRE RELATIONS FOR  $\mathfrak{k}$ 

The following three technical lemmas are used to derive the key equation (3.7). It is convenient to introduce the notation  $Q_X = \sum_{i \in X} \mathbb{Z}\alpha_i$  and  $Q_X^\pm := Q^+ \cap Q_X$ .

**Lemma A.1.** *Let  $(X, \tau) \in \text{CDec}(A)$  and  $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$ . For all  $i \in X$ ,  $j \in I$  and  $m \in \mathbb{Z}_{\geq 1}$  we have*

$$\text{ad}(b_i)^m(b_j) = \begin{cases} \text{ad}(f_i)^m(f_j) + \gamma_j \theta(\text{ad}(f_i)^m(f_j)) & \text{if } j \in I \setminus X, \\ \text{ad}(f_i)^m(f_j) & \text{if } j \in X. \end{cases}$$

*Proof.* This follows immediately from (2.7) and the fact that  $\theta$  is a Lie algebra automorphism.  $\square$

**Lemma A.2.** *Let  $(X, \tau) \in \text{CDec}(A)$  and  $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$ . For all  $i \in I \setminus X$ ,  $j \in X$  and  $m \in \mathbb{Z}_{\geq 1}$  we have*

$$\text{ad}(b_i)^m(b_j) = \text{ad}(f_i)^m(f_j) + \gamma_i^m \theta(\text{ad}(f_i)^m(f_j)) + \text{LO}_{ij}(m)$$

where

$$\text{LO}_{ij}(m) = \begin{cases} (1 + \zeta(\alpha_i))\gamma_i [\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^+ & \text{if } \tau(i) = i, w_X(\alpha_i) - \alpha_i - \alpha_j \in \Phi^+, m = 2, \\ -\gamma_i(2h_i - a_{ij}h_j) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 2, \\ -3(2 + a_{ij})\gamma_i(f_i - \theta(f_i)) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 3, \\ -6a_{ij}(2 + a_{ij})\gamma_i^2 e_j & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 4, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By induction with respect to  $m$ . For  $m = 1$ , (2.7) implies

$$\text{ad}(b_i)^1(b_j) = [f_i + \gamma_i \theta(f_i), f_j] = \text{ad}(f_i)^1(f_j) + \gamma_i^1 \theta(\text{ad}(f_i)^1(f_j)) + \text{LO}_{ij}(1)$$

with  $\text{LO}_{ij}(1) = 0$  as required. Now assume  $m \in \mathbb{Z}_{>1}$  and suppose the statement holds for all smaller values. Then, by virtue of the induction hypothesis, the fact that  $\theta$  is a Lie algebra automorphism and (2.7), we find

$$\begin{aligned} \text{ad}(b_i)^m(b_j) &= [b_i, \text{ad}(b_i)^{m-1}(b_j)] \\ &= [f_i + \gamma_i \theta(f_i), \text{ad}(f_i)^{m-1}(f_j) + \gamma_i^{m-1} \theta(\text{ad}(f_i)^{m-1}(f_j)) + \text{LO}_{ij}(m-1)] \\ &= \text{ad}(f_i)^m(f_j) + \gamma_i^m \theta(\text{ad}(f_i)^m(f_j)) \\ &\quad + \gamma_i [\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \gamma_i^{m-1} [f_i, \theta(\text{ad}(f_i)^{m-1}(f_j))] + [b_i, \text{LO}_{ij}(m-1)]. \end{aligned}$$

Using (2.8) we have  $\theta^2(f_i) = \zeta(\alpha_i)f_i$  so that

$$(A.1) \quad \text{LO}_{ij}(m) = \gamma_i [\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \zeta(\alpha_i)\gamma_i^{m-1} \theta([\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)]) + [b_i, \text{LO}_{ij}(m-1)].$$

Suppose that  $[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] \neq 0$ . Then  $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi \cup \{0\}$ . Now  $\Phi = \Phi^+ \cup \Phi^-$  implies that  $\tau(i) = i$  and  $j \in \check{X}(i)$ .

If  $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^+$  we must have  $\tau(i) = i$  and  $m = 2$ ; because  $w_X(\alpha_{\tau(i)}) - \alpha_i - \alpha_j \in Q_X^+$  it follows that  $[\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^+$ . The claimed expression for  $\text{LO}_{ij}(2)$  follows immediately from (A.1); those for  $\text{LO}_{ij}(m)$  with  $m > 2$  from (3.2).

Now suppose  $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^- \cup \{0\}$ . Then  $\tau(i) = i$  and  $w_X(\alpha_i) \leq (m-1)\alpha_i + \alpha_j$  so that  $\check{X}(i) = \{j\}$  and hence  $a_{ji} < 0$ . In this case we readily obtain

$$w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j = (2-m)\alpha_i - (1+a_{ji})\alpha_j.$$

From  $\Phi = \Phi^+ \cup \Phi^-$  it follows that  $a_{ji} = -1$ . Now  $\mathbb{Z}\alpha_i \cap \Phi = \{\pm\alpha_i\}$  implies that  $m \in \{2, 3\}$ . We straightforwardly compute

$$[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] = \begin{cases} a_{ij}h_j - h_i & \text{if } m = 2, \\ -2(1+a_{ij})f_i & \text{if } m = 3, \end{cases}$$

and the claimed expressions for  $\text{LO}_{ij}(m)$  readily follow.  $\square$

For  $i, j \in I$  and  $m, r \in \mathbb{Z}$  such that  $0 \leq r \leq \lfloor m/2 \rfloor$  define  $p_{ij}^{(r,m)} \in \mathbb{Z}$  by

$$(A.2) \quad p_{ij}^{(0,m)} = -1, \quad p_{ij}^{(\frac{m+1}{2}, m)} = 0, \quad p_{ij}^{(r,m+2)} = p_{ij}^{(r,m+1)} - (m+1)(m+a_{ij})p_{ij}^{(r-1,m)}.$$

**Lemma A.3.** *Let  $(X, \tau) \in \text{CDec}(A)$  and  $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$ . For all  $i, j \in I \setminus X$  such that  $i \neq j$  and  $m \in \mathbb{Z}_{\geq 0}$  we have*

$$\text{ad}(b_i)^m(b_j) = \text{ad}(f_i)^m(f_j) + \theta(\text{ad}(f_i)^m(f_j)) + \text{LO}_{ij}(m)$$

where

$$\text{LO}_{ij}(m) = \begin{cases} (\gamma_i + \zeta(\alpha_i)\gamma_j) [\theta(f_i), f_j] \in \mathfrak{n}_X^+ & \text{if } \tau(i) = j, w_X(\alpha_i) - \alpha_i \in \Phi^+, m = 1, \\ \gamma_j h_i - \gamma_i h_j & \text{if } \tau(i) = j, w_X(\alpha_i) = \alpha_i, m = 1, \\ 2((\gamma_j - a_{ij}\gamma_i)f_i - \gamma_i(\gamma_i - a_{ij}\gamma_j)e_j) & \text{if } \tau(i) = j, w_X(\alpha_i) = \alpha_i, m = 2, \\ \sum_{r=1}^{\lfloor m/2 \rfloor} p_{ij}^{(r,m)} \gamma_i^r \text{ad}(b_i)^{m-2r}(b_j) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* As in the proof of Lemma A.2 we apply induction with respect to  $m$ . For  $m = 0$  we have

$$\text{ad}(b_i)^0(b_j) = b_j = f_j + \gamma_j \theta(f_j) = \text{ad}(f_i)^0(f_j) + \gamma_i^0 \gamma_j \theta(\text{ad}(f_i)^0(f_j)) + \text{LO}_{ij}(0)$$

with  $\text{LO}_{ij}(0) = 0$  as required. Now assume  $m \in \mathbb{Z}_{>0}$  and suppose the statement holds for all smaller values. Then, by the induction hypothesis,

$$\begin{aligned} \text{ad}(b_i)^m(b_j) &= [b_i, \text{ad}(b_i)^{m-1}(b_j)] \\ &= [f_i + \gamma_i \theta(f_i), \text{ad}(f_i)^{m-1}(f_j) + \gamma_i^{m-1} \gamma_j \theta(\text{ad}(f_i)^{m-1}(f_j)) + \text{LO}_{ij}(m-1)]. \end{aligned}$$

Rearranging terms and using that  $\theta$  is a Lie algebra automorphism we obtain

$$\begin{aligned} \text{ad}(b_i)^m(b_j) &= \text{ad}(f_i)^m(f_j) + \gamma_i^{m-1} \gamma_j \theta(\text{ad}(f_i)^m(f_j)) + \text{LO}_{ij}(m) \quad \text{where} \\ \text{LO}_{ij}(m) &= \gamma_i [\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \gamma_i^{m-1} \gamma_j [f_i, \theta(\text{ad}(f_i)^{m-1}(f_j))] + [b_i, \text{LO}_{ij}(m-1)]. \end{aligned}$$

Using (2.8) we obtain

$$(A.3) \quad \text{LO}_{ij}(m) = \gamma_i [\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \zeta(\alpha_i) \gamma_i^{m-1} \gamma_j \theta([\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)]) + [b_i, \text{LO}_{ij}(m-1)].$$

If  $[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] \neq 0$  then  $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi \cup \{0\}$ .

If  $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^+$  we must have  $j = \tau(i)$ ,  $\check{X}(i) \neq \emptyset$ ,  $m = 1$ ; since  $w_X(\alpha_{\tau(i)}) - \alpha_j \in Q_X^+$  it follows that  $[\theta(f_i), f_j] \in \mathfrak{n}_X^+$ . The expression for  $\text{LO}_{ij}(1)$  follows from (A.3);  $\text{LO}_{ij}(m) = 0$  with  $m > 1$  is a consequence of (3.2).

Now suppose  $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^- \cup \{0\}$ . It follows that  $\check{X}(i) = \emptyset$ , so  $\zeta(\alpha_i) = 1$ , and  $\tau(i) \in \{i, j\}$ . If  $\tau(i) = j$  then  $\mathbb{Z}\alpha_i \cap \Phi = \{\pm\alpha_i\}$  implies that  $m \in \{1, 2\}$ . Furthermore,  $\theta(f_i) = -e_j$  and  $a_{ij} = a_{ji}$ . Now (A.3) implies, as required,  $\text{LO}_{ij}(1) = \gamma_j h_i - \gamma_i h_j$ ,

$$\begin{aligned} \text{LO}_{ij}(2) &= \gamma_i \gamma_j \theta([-e_j, [f_i, f_j]]) + \gamma_i [-e_j, [f_i, f_j]] + [b_i, \text{LO}_{ij}(1)] \\ &= \gamma_i \gamma_j \theta([h_j, f_i]) + \gamma_i [h_j, f_i] + [\gamma_i h_j - \gamma_j h_i, f_i - \gamma_i e_j] \\ &= 2((\gamma_j - a_{ij}\gamma_i)f_i - \gamma_i(\gamma_i - a_{ij}\gamma_j)e_j) \end{aligned}$$

and  $\text{LO}_{ij}(m) = 0$  if  $m > 2$ .

It remains to deal with the case  $\check{X}(i) = \emptyset$  and  $\tau(i) = i$ , in which case  $\theta(f_i) = -e_i$ . A straightforward computation gives

$$[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] = (m-1)(m-2+a_{ij})\text{ad}(f_i)^{m-2}(f_j).$$

By virtue of the induction hypothesis, (A.3) simplifies to

$$\text{LO}_{ij}(m) = (m-1)(m-2+a_{ij})\gamma_i (\text{ad}(b_i)^{m-2}(b_j) - \text{LO}_{ij}(m-2)) + [b_i, \text{LO}_{ij}(m-1)],$$

from which the recursion (A.2) follows straightforwardly.  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, YORK, YO10 5DD, UK AND  
 INSTITUTE OF THEORETICAL PHYSICS AND ASTRONOMY, VILNIUS UNIVERSITY, SAULĖTEKIO AV. 3, VILNIUS 10257, LITHUANIA.  
 E-mail address: [vidas.regelskis@gmail.com](mailto:vidas.regelskis@gmail.com)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, YORK, YO10 5DD, UK AND  
 DEPARTMENT OF MATHEMATICS, HERIOT-WATT UNIVERSITY, EDINBURGH, EH14 4AS, UK  
 E-mail address: [b.vlaar@hw.ac.uk](mailto:b.vlaar@hw.ac.uk)