Badly approximable points on curves are winning

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Let \( r = (r_1, \ldots, r_n) \) be an \( n \)-tuple of weights, \( I \subset \mathbb{R} \) be an interval and \( \varphi : I \to \mathbb{R}^n \) be a nondegenerate analytic map. Then the set

\[
\{ x \in I : \varphi(x) \text{ is } r\text{-badly approximable} \}
\]

is absolute winning on \( I \).

An analytic map \( \varphi = (\varphi_1, \ldots, \varphi_n) : I \to \mathbb{R}^n \) is nondegenerate if 1, \( \varphi_1, \ldots, \varphi_n \) are linearly independent over \( \mathbb{R} \).

For example, 1, \( x, x^2, \ldots, x^n \).
Theorem (Dirichlet, 1842)

For any \( x \in \mathbb{R}^n \setminus \mathbb{Q}^n \) there are infinitely many \( p/q \in \mathbb{Q}^n \) such that

\[
\max_{1 \leq i \leq n} \left| x_i - \frac{p_i}{q} \right| < \frac{1}{q^{1+1/n}}.
\]

Definition (Badly approximable points)

\( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \mathbb{Q}^n \) is called badly approximable \( (x \in \text{Bad}_n) \) if

\[
\exists \ c > 0 \quad \max_{1 \leq i \leq n} \left| x_i - \frac{p_i}{q} \right| \geq \frac{c}{q^{1+1/n}} \quad \text{for all } q \in \mathbb{N} \text{ and } p, \ldots, p_n \in \mathbb{Z}.
\]

- Real algebraic numbers of degree 2 are in \( \text{Bad}_1 \).
- \( e \not\in \text{Bad}_1 \).
- It is not known if \( \text{Bad}_1 \) contains any other algebraic numbers.
- \( \text{Leb}(\text{Bad}_1) = 0. \) \( (\text{Borel 1909}) \quad \dim_H \text{Bad}_1 = 1. \) \( (\text{Jarník 1928}) \)
- If \( \deg_{\mathbb{Q}}(\alpha) = n + 1 \) then \( \alpha \in \text{Bad}_n \). \( (\text{Perron 1921}) \)
- \( \text{Bad}_n \) is uncountable \( (\text{Davenport 1964}) \)
**The dimension of Bad\( (n) \)**

**Theorem (W.M. Schmidt, 1966)**

\[
\dim \text{Bad}(n) = n.
\]

*Furthermore, Bad\( (n) \) is winning.*

**Schmidt’s Game** involves 2 players: Alice and Bob, and 0 < \( \alpha, \beta < 1 \). To start the game Bob chooses a closed ball \( B_0 \) of radius \( r(B_0) > 0 \). Then for \( n = 0, 1, 2, \ldots \), Alice chooses a closed ball \( A_{n+1} \subset B_n \) of radius \( \alpha r(B_n) \), and then Bob chooses a closed ball \( B_{n+1} \subset A_{n+1} \) of radius \( \beta r(A_{n+1}) \).

**Definition**

\( S \subset \mathbb{R}^n \) is \( \alpha \)-winning if Alice has a strategy to ensure that \( \bigcap_{n=1}^{\infty} B_n \subset S \ \forall \beta \).

**Properties of Schmidt’s winning sets:**

- Any winning set has full Hausdorff dimension.
- An affine transformation of a winning set is a winning set.
- A countable intersection of \( \alpha \)-winning sets is \( \alpha \)-winning.
Definition (Weighted Badly approximable points)

Let \( \mathbf{r} = (r_1, \ldots, r_n), r_i \geq 0 \) and \( r_1 + \cdots + r_n = 1 \) (\( \mathbf{r} \) is called \textit{weights}). The point \( \mathbf{x} \in \mathbb{R}^n \) is called \( \mathbf{r} \)-badly approximable (\( \mathbf{x} \in \text{Bad}(\mathbf{r}) \)) if \( \exists \ c > 0 \) such that

\[
\max_{1 \leq i \leq n} q^{1+r_i} \left| \frac{p_i}{q} \right| \geq c \quad \text{for all } q \in \mathbb{N} \text{ and } p_i \in \mathbb{Z}.
\]

\[
\bigcap_{r_1 + r_2 = 1} \text{Bad}(r_1, r_2) = \emptyset \quad \text{implies \ Littlewood’s conjecture.}
\]

Conjecture (Littlewood, 1930)

Let \( (x_1, \ldots, x_n) \in \mathbb{R}^n, n \geq 2 \). Then for any \( \varepsilon > 0 \) there are infinitely many \( q \in \mathbb{N} \) and \( p_1, \ldots, p_n \in \mathbb{Z} \) such that

\[
\left| \frac{p_1}{q} \right| \cdots \left| \frac{p_n}{q} \right| < \varepsilon q^{-n-1} = \prod_{i=1}^{n} \left( \frac{\varepsilon^{1/n}}{q^{1+r_i}} \right).
\]

Conjecture (Schmidt, 1982)

Let \( n = 2 \). Then \( \text{Bad}(\frac{1}{3}, \frac{2}{3}) \cap \text{Bad}(\frac{2}{3}, \frac{1}{3}) \neq \emptyset. \)
The dimension of $\text{Bad}(r)$

Kristensen, Thorn, Velani (2006) \[ \Rightarrow \dim \text{Bad}(r) = n. \]
Kleinbock, Weiss (2010)

use modified Schmidt’s game: the metric on $\mathbb{R}^n$ depends on $r$. This does not imply that $\text{Bad}(r_1) \cap \text{Bad}(r_2)$ is non-empty and therefore does not prove

Schmidt’s conjecture (1982):

$$\text{Bad}(\frac{1}{3}, \frac{2}{3}) \cap \text{Bad}(\frac{2}{3}, \frac{1}{3}) \neq \emptyset.$$ 

Theorem (Badziahin, Pollington and Velani, 2011)

Let $n = 2$, $\theta \in \text{Bad}$, $C_\theta = \{(x, y) \in \mathbb{R}^2 : x = \theta\}$ and the weights $r_k = (r_{k,1}, r_{k,2})$ satisfy $\lim \inf_{k \to \infty} \min\{r_{k,1}, r_{k,2}\} > 0$. Then

$$\dim \bigcap_{k=1}^{\infty} \text{Bad}(r_k) \cap C_\theta = 1.$$
Bad(r), \( n = 2 \)

Let \( n = 2 \) and \( C_\theta = \{(x, y) \in \mathbb{R}^2 : x = \theta\} \).

**An (2013):** If \( \theta \in \text{Bad} \) then \( \text{Bad}(r) \cap C_\theta \) is \( \frac{1}{2} \)-winning in \( C_\theta \). In particular, for any countable collection of weights \( r_k \), \( \dim(\bigcap_k \text{Bad}(r_k) \cap C_\theta) = 1 \).

**Nesharim (2013):** If \( \theta \in \text{Bad} \) then for any countable collection of weights \( r_k \), the intersection \( \bigcap_k \text{Bad}(r_k) \cap C_\theta \neq \emptyset \).

**Nesharim, Weiss (2013):** If \( \theta \in \text{Bad} \) then \( \text{Bad}(r) \cap C_\theta \) is absolute winning in \( C_\theta \). In particular, it is \( \frac{1}{2} \)-winning.

**An (2016):** \( \text{Bad}(r) \) is \( (24\sqrt{2})^{-1} \)-winning in \( \mathbb{R}^2 \).

**Nesharim, Simmons (2014):** Let \( n = 2 \). Then \( \text{Bad}(r) \) is hyperplane absolute winning, in particular, it is \( \frac{1}{2} \)-winning.

**Badziahin & Velani (2014):** For any planar curve \( C \) which is not a straight line \( \dim(\bigcap_k \text{Bad}(r_k) \cap C) = 1 \) provided that \( \lim_{k \to \infty} \min\{r_{k,1}, r_{k,2}\} > 0 \).

**An, B., Velani (2018):** For any planar curve \( C \) with non-zero curvature the projection of \( \text{Bad}(r_k) \cap C \) onto any axis is \( \frac{1}{2} \)-winning.

**Note:** The last two papers also deal with a class of straight lines.
Lemma

\( x \in \text{Bad}(r) \) if and only if \( \exists \ c > 0 \) such that for any \( a = (a_1, \ldots, a_n) \in \mathbb{Z}_\neq 0^n \) and \( a_0 \in \mathbb{Z} \) if \( |a_i| < H^{r_i} \) for all \( 1 \leq i \leq n \) then

\[
|a_0 + a_1 x_1 + \cdots + a_n x_n| \geq c H^{-1}.
\]

Remark: If \( \xi \in \mathbb{R} \) is such that \( (\xi, \xi^2, \ldots, \xi^n) \in \text{Bad}(n) \), then \( \xi \) is badly approximable by algebraic numbers of degree \( n \):

\[
\xi \in B_n = \left\{ x \in \mathbb{R} : \exists \ c_1 > 0 \text{ such that } |x - \alpha| \geq c_1 H(\alpha)^{-n-1} \right\}
\text{for all algebraic } \alpha, \deg \alpha \leq n.
\]

\( H(\alpha) \) denotes the (naive) height of \( \alpha \).

Recall the Wirsing–Schmidt problem (still open for \( n \geq 3 \)):

any transcendental \( \xi \in \mathbb{R} \) belongs to

\[
\mathcal{W}_n = \left\{ x \in \mathbb{R} : \exists \ c_2 > 0 \text{ such that } |x - \alpha| < c_2 H(\alpha)^{-n-1} \right\}
\text{for infinitely many algebraic } \alpha, \deg \alpha \leq n.
\]

Fact: If \( (\xi, \ldots, \xi^n) \in \text{Bad}(n) \), then \( \xi \in B_n \cap \mathcal{W}_n \).
Let $\varphi_\ell : I \to \mathbb{R}^{n_\ell}$ ($1 \leq \ell \leq L$) be non-degenerate and the weights $r_k$ ($k \in \mathbb{N}$) satisfy $\lim \inf_{k \to \infty} \min\{r_{k,1}, \ldots, r_{k,n}\} > 0$. Then

$$\dim \bigcap_{k=1}^{\infty} \bigcap_{\ell=1}^{L} \varphi_\ell^{-1}(\text{Bad}(r_k)) = 1.$$  \hfill (1)

Let $\varphi_\ell : I \to \mathbb{R}^{n_\ell}$ ($1 \leq \ell \leq L$) be non-degenerate. Then (1) holds for any sequence of weights $r_k$ ($k \in \mathbb{N}$).

Let $\varphi_\ell : I \to \mathbb{R}^{n_\ell}$ be any sequence of analytic non-degenerate maps. Then for any sequence of weights $r_k$ ($k \in \mathbb{N}$)

$$\dim \bigcap_{k=1}^{\infty} \bigcap_{\ell=1}^{L} \varphi_\ell^{-1}(\text{Bad}(r_k)) = 1.$$  \hfill (2)
Main Theorem

Theorem (B., Nesharim, Yang, 2020+)

Let \( r = (r_1, \ldots, r_n) \) be an \( n \)-tuple of weights, \( I \subset \mathbb{R} \) be an interval and \( \varphi : I \to \mathbb{R}^n \) be a nondegenerate analytic map. Then the set

\[
\{ x \in I : \varphi(x) \text{ is } r\text{-badly approximable} \}
\]

is absolute winning on \( I \).

- any absolute winning set is \( \frac{1}{2} \)-winning;
- absolute winning sets are preserved by-Lipschitz maps and \( C^1 \) diffeomorphisms;
- any countable intersection of absolute winning sets is absolute winning;
- any absolute winning set has full Hausdorff dimension;
- for any absolute winning set \( S \), any open set \( U \) and any Ahlfors regular measure \( \mu \) such that \( U \cap \text{supp } \mu \neq \emptyset \) we have that

\[
\dim(S \cap U \cap \text{supp } \mu) = \dim(\text{supp } \mu).
\]
The Absolute Game (McMullen 2010) is played by Alice and Bob. Bob chooses $0 < \beta < 1$ and a closed ball $B_0$ of radius $r(B_0) > 0$. Then for $n = 0, 1, 2, \ldots$, Alice chooses a closed ball $A_{n+1}$ of radius $\beta r(B_n)$, then Bob chooses a closed ball $B_{n+1} \subset B_n \setminus A_{n+1}$ of radius $\beta r(B_n)$.

**Definition**

$S \subset \mathbb{R}^n$ is **absolute winning** if Alice has a strategy to ensures that either for some $n$ Bob cannot choose a ball $B_{n+1}$, or $\bigcap_{n=1}^{\infty} B_n \subset S$.

**Lemma (Badziahin, Harrap, Nesharim, Simmons, 2018, arXiv:1804.06499)**

Let $S \subset \mathbb{R}$ be a Borel subset. Suppose that $S \cap \text{supp} \mu \neq \emptyset$ for every Ahlfors regular measure $\mu$. Then $S$ is absolute winning.

A Borel measure $\mu$ on $\mathbb{R}^d$ is **Ahlfors regular** if there exist $C, \alpha, \rho_0 > 0$ such that for any ball $B(x, \rho) \subset \mathbb{R}$ with $x \in \text{supp} \mu$ and $\rho \leq \rho_0$ we have that

$$C^{-1} \rho^\alpha \leq \mu(B(x, \rho)) \leq C \rho^\alpha.$$
Theorem (B., Nesharim, Yang, 2020+)

Let \( r = (r_1, \ldots, r_n) \) be an \( n \)-tuple of weights, \( I \subset \mathbb{R} \) be an interval and \( \varphi : I \to \mathbb{R}^n \) be a nondegenerate analytic map. Then the set

\[
\varphi^{-1}(\text{Bad}(r)) = \{ x \in I : \varphi(x) \text{ is } r\text{-badly approximable} \}
\]

is absolute winning on \( I \).

How do we prove this? By finding a suitable non-empty Cantor set inside this intersection.
Generalised Cantor sets of Badziahin and Valani

Let $R \in \mathbb{Z}$, $R \geq 2$ and $\mathcal{I} \mapsto \text{Par}_R(\mathcal{I})$ be a map that divides each interval in the collection $\mathcal{I}$ into $R$ equal closed subintervals.

**Example:** $R = 3$ and $\mathcal{I} = \{[0, 1]\}$

$$\mathcal{I} = \{[0, 1]\}$$

$$\text{Par}_3(\mathcal{I}) = \{[0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1]\}$$

$$\text{Par}_3^2(\mathcal{I}) = \{[0, \frac{1}{9}], [\frac{1}{9}, \frac{2}{9}], \ldots\}$$

$\mathcal{J}_0 = \{I_0\}$. Then for $q = 0, 1, 2, \ldots$

- **Splitting:** $\mathcal{I}_{q+1} := \text{Par}_R(\mathcal{J}_q)$.
- **Removal:** $\mathcal{J}_{q+1} := \mathcal{I}_{q+1} \setminus \hat{\mathcal{J}_q}$.

**Example:** the middle third Cantor set construction

$\mathcal{K}(\mathcal{I}_q) \overset{\text{def}}{=} \bigcap_{q \geq 0} \bigcup_{l_q \in \mathcal{I}_q} l_q$
Let $h = (h_{p,q})_{0 \leq p \leq q}$ a sequence of non-negative integers.

For $q \geq 0$ write $\hat{J}_q$ as the following union

$$\hat{J}_q = \bigcup_{p=0}^{q} \hat{J}_{p,q}.$$

If for any $0 \leq p \leq q$ we have that

$$\# \{ I \in \hat{J}_{p,q} : I \subset J \} \leq h_{p,q} \text{ for all } J \in J_p,$$

then $K_\infty$ is called an $(R,h)$-Cantor set.

**Theorem (Badziahin, Velani 2011)**

Given an integer $R \geq 2$ and a sequence of non-negative integers $h = (h_{p,q})_{0 \leq p \leq q}$, let $t_0 = R - h_{0,0}$ and

$$t_q := R - h_{q,q} - \sum_{j=1}^{q} \frac{h_{q-j,q}}{\prod_{i=1}^{j} t_{q-i}} \quad \text{for } q \geq 1.$$

Suppose that $t_q > 0$ for all $q \geq 0$. Then every $(R,h)$-Cantor set is nonempty.
Dani’s correspondence

\[ X_{n+1} = \frac{\text{SL}_{n+1}(\mathbb{R})}{\text{SL}_{n+1}(\mathbb{Z})} \] – the space of unimodular lattices in \( \mathbb{R}^{n+1} \):

\[ g \in \text{SL}_{n+1}(\mathbb{R}) \mapsto g\mathbb{Z}^{n+1} \in X_{n+1}. \]

Mahler’s compactness theorem: \( S \subset X_{n+1} \) is bounded iff \( \exists \, \varepsilon > 0 \) s.t.

\[ S \subset K_\varepsilon := \left\{ \Lambda \in X_{n+1} : \inf_{\mathbf{v} \in \Lambda, \mathbf{v} \neq 0} \| \mathbf{v} \| \geq \varepsilon \right\} \]

Given \( x \in \mathbb{R}^n \), define the matrix

\[ u(x) := \begin{bmatrix} 1 & x \\ \mathbb{I}_n \end{bmatrix} \in \text{SL}_{n+1}(\mathbb{R}), \quad (3) \]

where \( \mathbb{I}_n \) is the identity matrix. For \( t \in \mathbb{R} \) let

\[ a(t) := \text{diag} \{ e^t, e^{-r_1 t}, \ldots, e^{-r_n t} \}. \]

**Lemma (Dani’s correspondence)**

Let \( x \in \mathbb{R}^n \). Then, \( x \in \text{Bad}(r) \iff \{ a(t)u(x)\mathbb{Z}^{n+1} : t > 0 \} \) is bounded.
Using the gradient

\[ \varphi(x) = (x, \varphi_2(x), \ldots, \varphi_n(x)) \]

\[ r_1 \geq \cdots \geq r_n > 0. \]

\[ z(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \varphi'_2(x) & \cdots & \varphi'_n(x) \\ 1 & 1 & \cdots & 1 \end{bmatrix} \]

**Fact:** \( a(t)u(\varphi(x))\mathbb{Z}^{n+1} \) is bounded \( \iff \) \( a(t)z(x)u(\varphi(x))\mathbb{Z}^{n+1} \) is bounded.

\[ z(x)u(\varphi(x)) = \begin{bmatrix} 1 & x & \varphi_2(x) & \cdots & \varphi_n(x) \\ 1 & \varphi'_2(x) & \cdots & \varphi'_n(x) \\ 1 & 1 & \cdots & 1 \end{bmatrix} \]

\[ b(l) := \text{diag} \left\{ e^{-l/n}, e^l, e^{-l/n}, \ldots, e^{-l/n} \right\} . \]
\[ J_0 = \{ l_0 \}. \] For \( q = 0, 1, 2, \ldots \):

\[ I_{q+1} := \text{Par}_R(J_q), \quad J_{q+1} := I_{q+1} \setminus \hat{J}_{q}, \quad \hat{J}_q = \bigcup_{p=0}^{q} \hat{J}_{p,q}. \]

**Parameters:** large integer \( R \) and \( \beta, \beta' > 1 \) defined by

\[ e^{(1+r_1)\beta} = R \quad \text{and} \quad e^{(1+1/n)\beta'} = R. \]

\[ \hat{J}_{q,q} := \{ I \in I_{q+1} : \mu(I) < (3C)^{-1} ||I||_\alpha \}. \] (4)

For \( p < q \) we use the following **condition**:

\[ b(\beta'l) a(\beta(q+1))z(x)u(\varphi(x))\mathbb{Z}^{n+1} \not\subseteq K_{e^{-\epsilon\beta l}} \] for some \( x \in I \). (5)

Specifically,

\[ \hat{J}_{0,q} := \left\{ I \in I_{q+1} \setminus \hat{J}_{q,q} : \exists l \in \mathbb{N}, \ q/8 \leq l \leq q/4 \text{ s.t. (5) holds} \right\} \]

and for \( q/2 < p < q \) with \( p = q - 4l \) for some \( l \in \mathbb{Z} \) define

\[ \hat{J}_{p,q} := \left\{ I \in I_{q+1} \setminus \left( \hat{J}_{q,q} \cup \bigcup_{0 \leq p' < p} \hat{J}_{p',q} \right) : (5) \text{ holds} \right\}. \]
Proposition

If $\mu$ is an $\alpha$-Ahlfors regular measure and $I_0 \subset I$ is a sufficiently small interval centred in $\text{supp } \mu$, then for all sufficiently large $R$ the Cantor set defined on the previous slide is an $(R, h)$-Cantor set (in the sense of Badziahin-Velani) with

$$h_{q,q} \leq R - (4C)^{-2} R^\alpha, \quad h_{p,q} \leq C_1 R^{\alpha (1-\eta)(q+1-p)} \quad (0 \leq p < q),$$

where $\eta > 0$, $C > 0$ depends only of $\mu$ and $C_1$ is independent of $R$.

Theorem (Badziahin, Velani 2011)

Given an integer $R \geq 2$ and a sequence of non-negative integers $h = (h_{p,q})_{0 \leq p \leq q}$, let $t_0 = R - h_{0,0}$ and

$$t_q := R - h_{q,q} - \sum_{j=1}^{q} \frac{h_{q-j,q}}{\prod_{i=1}^{j} t_{q-i}} \quad \text{for } q \geq 1.$$

If $t_q > 0$ for all $q \geq 0$, then every $(R, h)$-Cantor set is nonempty.
Theorem (Badziahin, Velani 2011)

Given an integer $R \geq 2$ and a sequence of non-negative integers $h = (h_{p,q})_{0 \leq p \leq q}$, let $t_0 = R - h_{0,0}$ and

$$t_q := R - h_{q,q} - \sum_{j=1}^{q} \frac{h_{q-j,q}}{\prod_{i=1}^{j} t_{q-i}}$$

for $q \geq 1$.

If $t_q > 0$ for all $q \geq 0$, then every $(R, h)$-Cantor set is nonempty.

Corollary (Follows from the proposition)

For $R$ sufficiently large

$$t_q \geq (6C)^{-2}R^\alpha > 0 \quad \text{for all } q.$$

Hence $\mathcal{K}_\infty \neq \emptyset$.

Recall: $\mathcal{K}_\infty \subset \varphi^{-1}(\text{Bad}(r)) \cap \text{supp } \mu$. Hence, the corollary completes the proof of the main theorem.
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