Badly approximable points on curves are winning

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(joint work with **Erez Nesharim** and **Lei Yang**)

(arXiv:2005.02128)

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Destination of today's talk

Theorem (B., Nesharim, Yang, 2020, arXiv:2005.02128)

Let $\mathbf{r} = (r_1, \ldots, r_n)$ *be an n-tuple of weights,* $I \subset \mathbb{R}$ *be an interval and* $\varphi : I \to \mathbb{R}^n$ *be a nondegenerate analytic map. Then the set*

$$\{x \in I : \varphi(x) \textit{ is } \mathbf{r}\textit{-badly approximable}\}$$

is absolute winning on I.

An analytic map $\varphi = (\varphi_1, \ldots, \varphi_n) : I \to \mathbb{R}^n$ is *nondegenerate* if $1, \varphi_1, \ldots, \varphi_n$ are linearly independent over $\mathbb{R}$.
For example, $1, x, x^2, \ldots, x^n$.

Bad definition & early history

Theorem (Dirichlet, 1842)

For any $\mathbf{x} \in \mathbb{R}^n \setminus \mathbb{Q}^n$ *there are infinitely many* $\mathbf{p}/q \in \mathbb{Q}^n$ *such that*

$$\max_{1 \le i \le n} \left| x_i - \frac{p_i}{q} \right| < \frac{1}{q^{1+1/n}}.$$

Definition (Badly approximable points)

$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \mathbb{Q}^n$ is called badly approximable ($\mathbf{x} \in \mathbf{Bad}_n$) if $\exists\ c > 0$

$$\max_{1 \le i \le n} \left| x_i - \frac{p_i}{q} \right| \ge \frac{c}{q^{1+1/n}} \quad \text{for all } q \in \mathbb{N} \text{ and } p, \ldots, p_n \in \mathbb{Z}.$$

- Real algebraic numbers of degree 2 are in $\mathbf{Bad}_1$.
- $e \notin \mathbf{Bad}_1$.
- ? It is not known if $\mathbf{Bad}_1$ contains any other algebraic numbers.
- $\mathrm{Leb}(\mathbf{Bad}_1) = 0$. **(Borel 1909)** $\dim_{\mathrm{H}} \mathbf{Bad}_1 = 1$. **(Jarník 1928)**
- If $\deg_{\mathbb{Q}}(\alpha) = n + 1$ then $\alpha \in \mathbf{Bad}_n$. **(Perron 1921)**
- $\mathbf{Bad}_n$ is uncountable **(Davenport 1964)**

The dimension of **Bad**(n)

Theorem (W.M. Schmidt, 1966)

$$\dim \mathbf{Bad}(n) = n.$$

Furthermore, **Bad**(n) *is winning.*

Schmidt's Game involves 2 players: Alice and Bob, and $0 < \alpha, \beta < 1$. To start the game Bob chooses a closed ball B_0 of radius $r(B_0) > 0$.Then for $n = 0, 1, 2, \ldots$,Alice chooses a closed ball $A_{n+1} \subset B_n$ of radius $\alpha r(B_n)$,and then Bob chooses a closed ball $B_{n+1} \subset A_{n+1}$ of radius $\beta r(A_{n+1})$.

Definition

$S \subset \mathbb{R}^n$ is α-*winning* if Alice has a strategy to ensures $\bigcap_{n=1}^{\infty} B_n \subset S \ \forall \beta$.

Properties of Schmidt's winning sets:

- Any winning set has full Hausdorff dimension.
- An affine transformation of a wining set is a winning set.
- A countable intersection of α-winning sets is α-winning.

Bad with weights: Schmidt and Littlewood

Definition (Weighted Badly approximable points)

Let $\mathbf{r} = (r_1, \ldots, r_n)$, $r_i \geq 0$ and $r_1 + \cdots + r_n = 1$ ($\mathbf{r}$ is called *weights*). The point $\mathbf{x} \in \mathbb{R}^n$ is called $\mathbf{r}$-badly approximable ($\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$) if $\exists\ c > 0$ such that

$$\max_{1 \leq i \leq n} q^{1+r_i} \left| x_i - \frac{p_i}{q} \right| \geq c \quad \text{for all } q \in \mathbb{N} \text{ and } p_i \in \mathbb{Z}.$$

$\bigcap_{r_1+r_2=1} \mathbf{Bad}(r_1, r_2) = \emptyset$ implies Littlewood's conjecture.

Conjecture (Littlewood, 1930)

Let $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$. *Then for any* $\varepsilon > 0$ *there are infinitely many* $q \in \mathbb{N}$ *and* $p_1, \ldots, p_n \in \mathbb{Z}$ *such that*

$$\left| x_1 - \frac{p_1}{q} \right| \cdots \left| x_n - \frac{p_n}{q} \right| < \varepsilon q^{-n-1} = \prod_{i=1}^{n} \left(\frac{\varepsilon^{1/n}}{q^{1+r_i}} \right).$$

Conjecture (Schmidt, 1982)

Let $n = 2$. *Then* $\mathbf{Bad}(\frac{1}{3}, \frac{2}{3}) \cap \mathbf{Bad}(\frac{2}{3}, \frac{1}{3}) \neq \emptyset$.

The dimension of $\mathbf{Bad}(\mathbf{r})$

Kristensen, Thorn, Velani (2006)
Kleinbock, Weiss (2010) $\quad \Rightarrow \quad \dim \mathbf{Bad}(\mathbf{r}) = n.$

$\nwarrow$

use modified Schmidt's game: the metric on $\mathbb{R}^n$ depends on $\mathbf{r}$.
This does not imply that $\mathbf{Bad}(\mathbf{r}_1) \cap \mathbf{Bad}(\mathbf{r}_2)$ is non-empty and therefore does not prove

Schmidt's conjecture (1982):

$$\mathbf{Bad}(\tfrac{1}{3}, \tfrac{2}{3}) \cap \mathbf{Bad}(\tfrac{2}{3}, \tfrac{1}{3}) \neq \emptyset.$$

Theorem (Badziahin, Pollington and Velani, 2011)

Let $n = 2$, $\theta \in \mathbf{Bad}$, $\mathcal{C}_\theta = \{(x, y) \in \mathbb{R}^2 : x = \theta\}$ *and the weights* $\mathbf{r}_k = (r_{k,1}, r_{k,2})$ *satisfy* $\liminf_{k\to\infty} \min\{r_{k,1}, r_{k,2}\} > 0$. *Then*

$$\dim \bigcap_{k=1}^{\infty} \mathbf{Bad}(\mathbf{r}_k) \cap \mathcal{C}_\theta = 1\,.$$

$\mathbf{Bad}(\mathbf{r})$, $n = 2$

Let $n = 2$ and $\mathcal{C}_\theta = \{(x, y) \in \mathbb{R}^2 : x = \theta\}$.

An (2013): If $\theta \in \mathbf{Bad}$ then $\mathbf{Bad}(\mathbf{r}) \cap \mathcal{C}_\theta$ is $\frac{1}{2}$-winning in $\mathcal{C}_\theta$. In particular, for any countable collection of weights $\mathbf{r}_k$, $\dim(\bigcap_k \mathbf{Bad}(\mathbf{r}_k) \cap \mathcal{C}_\theta) = 1$.

Nesharim (2013): If $\theta \in \mathbf{Bad}$ then for any countable collection of weights $\mathbf{r}_k$, the intersection $\bigcap_k \mathbf{Bad}(\mathbf{r}_k) \cap \mathcal{C}_\theta \neq \emptyset$.

Nesharim, Weiss (2013): If $\theta \in \mathbf{Bad}$ then $\mathbf{Bad}(\mathbf{r}) \cap \mathcal{C}_\theta$ is absolute winning in $\mathcal{C}_\theta$. In particular, it is $\frac{1}{2}$-winning.

An (2016): $\mathbf{Bad}(\mathbf{r})$ is $(24\sqrt{2})^{-1}$-winning in $\mathbb{R}^2$.

Nesharim, Simmons (2014): Let $n = 2$. Then $\mathbf{Bad}(\mathbf{r})$ is hyperplane absolute winning, in particular, it is $\frac{1}{2}$-winning.

Badziahin & Velani (2014): For any planar curve $\mathcal{C}$ which is not a straight line $\dim(\bigcap_k \mathbf{Bad}(\mathbf{r}_k) \cap \mathcal{C}) = 1$ provided that $\liminf_{k\to\infty} \min\{r_{k,1}, r_{k,2}\} > 0$.

An, B., Velani (2018): For any planar curve $\mathcal{C}$ with non-zero curvature the projection of $\mathbf{Bad}(\mathbf{r}_k) \cap \mathcal{C}$ onto any axis is $\frac{1}{2}$-winning.

Note: The last two papers also deal with a class of straight lines.

Dual form

Lemma

$\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$ *if and only if* $\exists\ c > 0$ *such that for any* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\neq \mathbf{0}}$ *and* $a_0 \in \mathbb{Z}$ *if* $|a_i| < H^{r_i}$ *for all* $1 \le i \le n$ *then*

$$|a_0 + a_1 x_1 + \cdots + a_n x_n| \ge cH^{-1}.$$

Remark: If $\xi \in \mathbb{R}$ is such that $(\xi, \xi^2, \ldots, \xi^n) \in \mathbf{Bad}(n)$, then ξ is badly approximable by algebraic numbers of degree n:

$$\xi \in \mathcal{B}_n = \left\{ x \in \mathbb{R} : \begin{array}{l} \exists\ c_1 > 0 \text{ such that } |x - \alpha| \ge c_1 H(\alpha)^{-n-1} \\ \text{for all algebraic } \alpha,\ \deg \alpha \le n. \end{array} \right\}.$$

$H(\alpha)$ denotes the (naive) height of α.
Recall the **Wirsing-Schmidt problem** (still open for $n \ge 3$):
any transcendental $\xi \in \mathbb{R}$ belongs to

$$\mathcal{W}_n = \left\{ x \in \mathbb{R} : \begin{array}{l} \exists\ c_2 > 0 \text{ such that } |x - \alpha| < c_2 H(\alpha)^{-n-1} \\ \text{for infinitely many algebraic } \alpha,\ \deg \alpha \le n. \end{array} \right\}.$$

Fact: If $(\xi, \ldots, \xi^n) \in \mathbf{Bad}(n)$, then $\xi \in \mathcal{B}_n \cap \mathcal{W}_n$.

$\mathbf{Bad}(\mathbf{r})$, arbitrary n

Theorem (B., 2015)

Let $\varphi_\ell : I \to \mathbb{R}^{n_\ell}$ $(1 \le \ell \le L)$ be non-degenerate and the weights $\mathbf{r}_k$ $(k \in \mathbb{N})$ satisfy $\liminf_{k\to\infty} \min\{r_{k,1}, \dots, r_{k,n}\} > 0$. Then

$$\dim \bigcap_{k=1}^{\infty} \bigcap_{\ell=1}^{L} \varphi_\ell^{-1}(\mathbf{Bad}(\mathbf{r}_k)) = 1. \tag{1}$$

Theorem (Yang, 2019)

Let $\varphi_\ell : I \to \mathbb{R}^{n_\ell}$ $(1 \le \ell \le L)$ be non-degenerate. Then (1) holds for any sequence of weights $\mathbf{r}_k$ $(k \in \mathbb{N})$.

Theorem (B., Nesharim, Yang, 2020+)

Let $\varphi_\ell : I \to \mathbb{R}^{n_\ell}$ be any sequence of analytic non-degenerate maps. Then for any sequence of weights $\mathbf{r}_k$ $(k \in \mathbb{N})$

$$\dim \bigcap_{k=1}^{\infty} \bigcap_{\ell=1}^{\infty} \varphi_\ell^{-1}(\mathbf{Bad}(\mathbf{r}_k)) = 1. \tag{2}$$

Main Theorem

Theorem (B., Nesharim, Yang, 2020+)

Let $\mathbf{r} = (r_1, \ldots, r_n)$ be an n-tuple of weights, $I \subset \mathbb{R}$ be an interval and $\varphi : I \to \mathbb{R}^n$ be a nondegenerate analytic map. Then the set

$$\{x \in I : \varphi(x) \text{ is } \mathbf{r}\text{-badly approximable}\}$$

is absolute winning on I.

- any absolute winning set is $\frac{1}{2}$-winning;
- absolute winning sets are preserved by-Lipschitz maps and C^1 diffeomorphisms;
- any countable intersection of absolute winning sets is absolute winning;
- any absolute winning set has full Hausdorff dimension;
- for any absolute winning set S, any open set U and any Ahlfors regular measure μ such that $U \cap \operatorname{supp} \mu \neq \emptyset$ we have that

$$\dim(S \cap U \cap \operatorname{supp} \mu) = \dim(\operatorname{supp} \mu) .$$

The Absolute Game

The Absolute Game (McMullen 2010) is played by Alice and Bob. Bob chooses $0 < \beta < 1$ and a closed ball B_0 of radius $r(B_0) > 0$. Then for $n = 0, 1, 2, \ldots$,Alice chooses a closed ball A_{n+1} of radius $\beta r(B_n)$, then Bob chooses a closed ball $B_{n+1} \subset B_n \setminus A_{n+1}$ of radius $\beta r(B_n)$.

Definition

$S \subset \mathbb{R}^n$ is *absolute winning* if Alice has a strategy to ensures that either for some n Bob cannot choose a ball B_{n+1}, or $\bigcap_{n=1}^{\infty} B_n \subset S$.

Lemma (Badziahin, Harrap, Nesharim, Simmons, 2018, arXiv:1804.06499)

Let $S \subset \mathbb{R}$ be a Borel subset. Suppose that $S \cap \operatorname{supp} \mu \neq \emptyset$ for every Ahlfors regular measure μ. Then S is absolute winning.

A Borel measure μ on $\mathbb{R}^d$ is *Ahlfors regular* if there exist $C, \alpha, \rho_0 > 0$ such that for any ball $B(x, \rho) \subset \mathbb{R}$ with $x \in \operatorname{supp} \mu$ and $\rho \leq \rho_0$ we have that

$$C^{-1} \rho^{\alpha} \leq \mu(B(x, \rho)) \leq C \rho^{\alpha} .$$

Intersections with fractals

Theorem (B., Nesharim, Yang, 2020+)

Let $\mathbf{r} = (r_1, \ldots, r_n)$ *be an n-tuple of weights,* $I \subset \mathbb{R}$ *be an interval and* $\varphi : I \to \mathbb{R}^n$ *be a nondegenerate analytic map. Then the set*

$$\varphi^{-1}(\mathbf{Bad}(\mathbf{r})) = \{x \in I : \varphi(x) \text{ is } \mathbf{r}\text{-badly approximable}\}$$

is absolute winning on I*.*

$$\Updownarrow$$

Theorem (B., Nesharim, Yang, 2020+)

Let $\mathbf{r} = (r_1, \ldots, r_n)$ *be an n-tuple of weights,* $I \subset \mathbb{R}$ *be an interval and* $\varphi : I \to \mathbb{R}^n$ *be a nondegenerate analytic map. Then for any Ahlfors regular measure* μ *with* $\operatorname{supp} \mu \cap I \neq \emptyset$

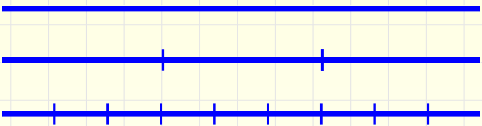
$$\operatorname{supp} \mu \cap \varphi^{-1}(\mathbf{Bad}(\mathbf{r})) \neq \emptyset.$$

How do we prove this? By finding a suitable non-empty Cantor set inside this intersection.

Generalised Cantor sets of Badziahin and Valani

Let $R \in \mathbb{Z}$, $R \geq 2$ and $\mathcal{I} \mapsto \mathbf{Par}_R(\mathcal{I})$ be a map that divides each interval in the collection $\mathcal{I}$ into R equal closed subintervals.

Example: $R = 3$ and $\mathcal{I} = \{[0,1]\}$



$\mathcal{I} = \{[0,1]\}$

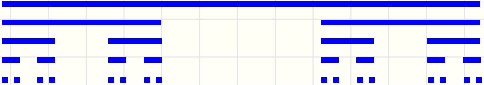
$\mathbf{Par}_3(\mathcal{I}) = \{[0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1]\}$

$\mathbf{Par}_{3^2}(\mathcal{I}) = \{[0, \frac{1}{9}], [\frac{1}{9}, \frac{2}{9}], \ldots$

$\mathcal{J}_0 = \{I_0\}$. Then for $q = 0, 1, 2, \ldots$

- **Splitting:** $\mathcal{I}_{q+1} := \mathbf{Par}_R(\mathcal{J}_q)$.
- **Removal:** $\mathcal{J}_{q+1} := \mathcal{I}_{q+1} \setminus \hat{\mathcal{J}}_q$.

Example: the middle third Cantor set construction



Limit points set: $\mathcal{K}(\mathcal{I}_q) \stackrel{\text{def}}{=} \bigcap_{q \geq 0} \bigcup_{I_q \in \mathcal{I}_q} I_q$

$(R, \mathbf{h})$-Cantor sets

Let $\mathbf{h} = (h_{p,q})_{0 \le p \le q}$ a sequence of non-negative integers.
For $q \ge 0$ write $\hat{\mathcal{J}}_q$ as the following union

$$\hat{\mathcal{J}}_q = \bigcup_{p=0}^{q} \hat{\mathcal{J}}_{p,q} \, .$$

If for any $0 \le p \le q$ we have that

$$\#\{I \in \hat{\mathcal{J}}_{p,q} : I \subset J\} \le h_{p,q} \quad \text{for all } J \in \mathcal{J}_p,$$

then $\mathcal{K}_\infty$ is called an $(R, \mathbf{h})$-*Cantor set*.

Theorem (Badziahin, Velani 2011)

Given an integer $R \ge 2$ and a sequence of non-negative integers $\mathbf{h} = (h_{p,q})_{0 \le p \le q}$, let $t_0 = R - h_{0,0}$ and

$$t_q := R - h_{q,q} - \sum_{j=1}^{q} \frac{h_{q-j,q}}{\prod_{i=1}^{j} t_{q-i}} \qquad \text{for } q \ge 1 \, .$$

Suppose that $t_q > 0$ for all $q \ge 0$. Then every $(R, \mathbf{h})$-Cantor set is nonempty.

Dani's correspondence

$X_{n+1} = \mathrm{SL}_{n+1}(\mathbb{R})/\mathrm{SL}_{n+1}(\mathbb{Z})$ – the space of unimodular lattices in $\mathbb{R}^{n+1}$:

$$g \in \mathrm{SL}_{n+1}(\mathbb{R}) \mapsto g\mathbb{Z}^{n+1} \in X_{n+1}.$$

Mahler's compactness theorem: $S \subset X_{n+1}$ is bounded iff $\exists\ \varepsilon > 0$ s.t.

$$S \subset K_\epsilon := \left\{ \Lambda \in X_{n+1} : \inf_{\mathbf{v}\in\Lambda,\, \mathbf{v}\neq\mathbf{0}} \|\mathbf{v}\| \geq \varepsilon \right\}$$

Given $\mathbf{x} \in \mathbb{R}^n$, define the matrix

$$u(\mathbf{x}) := \begin{bmatrix} 1 & \mathbf{x} \\ & \mathbb{I}_n \end{bmatrix} \in \mathrm{SL}_{n+1}(\mathbb{R}), \qquad (3)$$

where $\mathbb{I}_n$ is the identity matrix. For $t \in \mathbb{R}$ let

$$a(t) := \mathrm{diag}\left\{ e^t, e^{-r_1 t}, \ldots, e^{-r_n t} \right\}.$$

Lemma (Dani's correspondence)

Let $\mathbf{x} \in \mathbb{R}^n$. Then, $\mathbf{x} \in \mathbf{Bad}(\mathbf{r}) \iff \left\{ a(t)u(\mathbf{x})\mathbb{Z}^{n+1} : t > 0 \right\}$ is bounded.

Using the gradient

$\varphi(x) = (x, \varphi_2(x), \ldots, \varphi_n(x))$ $\qquad r_1 \geq \cdots \geq r_n > 0.$

$$z(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & \varphi_2'(x) & \cdots & \varphi_n'(x) \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Fact: $a(t)u(\varphi(x))\mathbb{Z}^{n+1}$ is bounded $\iff$ $a(t)z(x)u(\varphi(x))\mathbb{Z}^{n+1}$ is bounded.

$$z(x)u(\varphi(x)) = \begin{bmatrix} 1 & x & \varphi_2(x) & \cdots & \varphi_n(x) \\ & 1 & \varphi_2'(x) & \cdots & \varphi_n'(x) \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$b(l) := \mathrm{diag}\left\{e^{-l/n}, e^{l}, e^{-l/n}, \ldots, e^{-l/n}\right\}.$$

Main Construction

$\mathcal{J}_0 = \{I_0\}$. For $q = 0, 1, 2, \ldots$:

$$\mathcal{I}_{q+1} := \mathbf{Par}_R(\mathcal{J}_q), \qquad \mathcal{J}_{q+1} := \mathcal{I}_{q+1} \setminus \hat{\mathcal{J}}_q, \qquad \hat{\mathcal{J}}_q = \bigcup_{p=0}^{q} \hat{\mathcal{J}}_{p,q}.$$

Parameters: large integer R and $\beta, \beta' > 1$ defined by

$$e^{(1+r_1)\beta} = R \qquad \text{and} \qquad e^{(1+1/n)\beta'} = R.$$

$$\hat{\mathcal{J}}_{q,q} := \left\{I \in \mathcal{I}_{q+1} \ : \ \mu(I) < (3C)^{-1}|I|^{\alpha}\right\}. \tag{4}$$

For $p < q$ we use the following **condition**:

$$b\left(\beta' l\right) a(\beta(q+1)) z(x) u(\varphi(x)) \mathbb{Z}^{n+1} \notin K_{e^{-\epsilon\beta l}} \quad \text{for some } x \in I\,. \tag{5}$$

Specifically,

$$\hat{\mathcal{J}}_{0,q} := \left\{I \in \mathcal{I}_{q+1} \setminus \hat{\mathcal{J}}_{q,q} : \exists\ l \in \mathbb{N},\ q/8 \le l \le q/4 \text{ s.t. (5) holds}\right\}$$

and for $q/2 < p < q$ with $p = q - 4l$ for some $l \in \mathbb{Z}$ define

$$\hat{\mathcal{J}}_{p,q} := \left\{I \in \mathcal{I}_{q+1} \setminus \left(\hat{\mathcal{J}}_{q,q} \cup \textstyle\bigcup_{0 \le p' < p} \hat{\mathcal{J}}_{p',q}\right) \ : \ \text{(5) holds}\right\}.$$

Key bounds:

Proposition

If μ is an α-Ahlfors regular measure and $I_0 \subset I$ is a sufficiently small interval centred in $\operatorname{supp}\mu$*, then for all sufficiently large R the Cantor set defined on the previous slide is an $(R, \mathbf{h})$-Cantor set (in the sense of Badziahin-Velani) with*

$$h_{q,q} \leq R - (4C)^{-2} R^{\alpha}, \qquad h_{p,q} \leq C_1 R^{\alpha(1-\eta)(q+1-p)} \ \ (0 \leq p < q)\,,$$

where $\eta > 0$, $C > 0$ depends only of μ and C_1 is independent of R.

Theorem (Badziahin, Velani 2011)

Given an integer $R \geq 2$ and a sequence of non-negative integers $\mathbf{h} = (h_{p,q})_{0 \leq p \leq q}$, let $t_0 = R - h_{0,0}$ and

$$t_q := R - h_{q,q} - \sum_{j=1}^{q} \frac{h_{q-j,q}}{\prod_{i=1}^{j} t_{q-i}} \qquad \text{for } q \geq 1\,.$$

If $t_q > 0$ for all $q \geq 0$, then every $(R, \mathbf{h})$-Cantor set is nonempty.

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If $t_q > 0$ for all $q \geq 0$, then every $(R, \mathbf{h})$-Cantor set is nonempty.

Corollary (Follows from the proposition)

For R sufficiently large

$$t_q \geq (6C)^{-2} R^{\alpha} > 0 \qquad \text{for all } q.$$

Hence $\mathcal{K}_\infty \neq \emptyset$.

Recall: $\mathcal{K}_\infty \subset \varphi^{-1}(\mathbf{Bad}(\mathbf{r})) \cap \operatorname{supp} \mu$. Hence, the corollary completes the proof of the main theorem.

The End